Signed Numbers: Positive and Negative, Opposites
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The basic operations with real numbers are presented in this chapter. The concept of absolute value is discussed both geometrically and symbolically. The geometric presentation offers a visual understanding of the meaning of |x|. The symbolic presentation includes a literal explanation of how to use the definition. Negative exponents are developed, using reciprocals and the rules of exponents the student has already learned. Scientific notation is also included, using unique and real-life examples. Objectives of this module: be familiar with positive and negative numbers and with the concept of opposites.

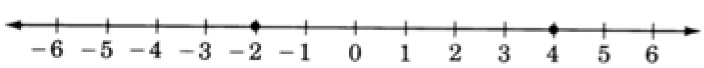
Overview

- Positive and Negative Numbers
- Opposites

Positive and Negative Numbers

When we studied the number line in Section [link] we noted that

Each point on the number line corresponds to a real number, and each real number is located at a unique point on the number line.



Positive and Negative Numbers
Each real number has a **sign** inherently associated with it. A real number is said to be a **positive number** if it is located to the right of 0 on the number line. It is a **negative** number if it is located to the left of 0 on the number line.

THE NOTATION OF SIGNED NUMBERS
A number is denoted as **positive** if it is directly preceded by a $''+''$ sign **or** no sign at all.
A number is denoted as **negative** if it is directly preceded by a $''-''$ sign.

The $''+''$ and $''-''$ signs now have two meanings:

$+$ can denote the operation of addition or a positive number.
$-$ can denote the operation of subtraction or a negative number.

Read the ”-”Sign as "Negative"
To avoid any confusion between "sign" and "operation," it is preferable to read the sign of a number as "positive" or "negative."

Sample Set A

Example:
-8 should be read as "negative eight" rather than "minus eight."

Example:
$4+(-2)$ should be read as "four plus negative two" rather than "four plus minus two."

Example:
$-6+(-3)$ should be read as "negative six plus negative three" rather than "minus six plusminus three."

Example:
$-15-(-6)$ should be read as "negative fifteen minus negative six" rather than "minus fifteenminus minus six."

Example:

$-5+7$ should be read as "negative five plus seven" rather than "minus five plus seven."

Example:
$0-2$ should be read as "zero minus two."

Practice Set A

Write each expression in words.

Exercise:

Problem: $4+10$

Solution:

four plus ten

Exercise:

Problem: $7+(-4)$

Solution:

seven plus negative four

Exercise:

Problem: $-9+2$

Solution:

negative nine plus two

Exercise:

Problem: $-16-(+8)$

Solution:

negative sixteen minus positive eight

Exercise:

Problem: $-1-(-9)$

Solution:

negative one minus negative nine

Exercise:

Problem: $0+(-7)$

Solution:

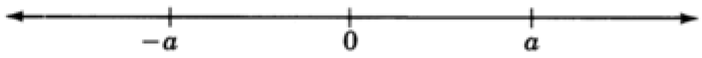
zero plus negative seven

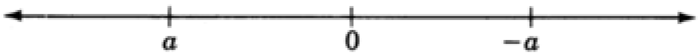
Opposites

Opposites
On the number line, each real number has an image on the opposite side of 0. For this reason we say that each real number has an opposite. **Opposites** are the same distance from zero but have opposite signs.

The opposite of a real number is denoted by placing a negative sign directly in front of the number. Thus, if a is any real number, then $-a$ is its opposite. **Notice** that the letter a is a variable. Thus, $''a''$ need not be positive, and $''-a''$ need not be negative.

If a is a real number, $-a$ is opposite a on the number line and a is opposite $-a$ on the number line.





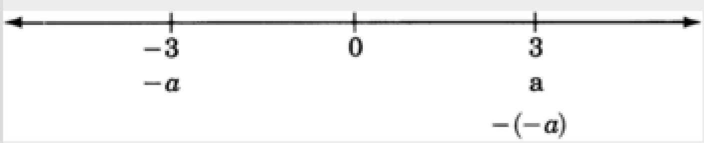
$-(-a)$ is opposite $-a$ on the number line. This implies that $-(-a) = a$.

This property of opposites suggests the double-negative property for real numbers.

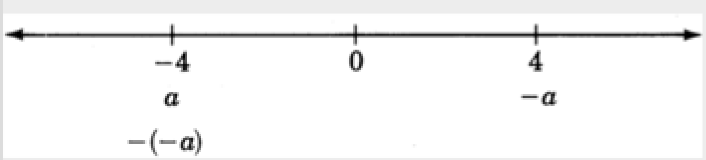
THE DOUBLE-NEGATIVE PROPERTY
If a is a real number, then
$-(-a) = a$

Sample Set B

Example:
If $a = 3$, then $-a = -3$ and $-(-a) = -(-3) = 3$.



Example:
If $a = -4$, then $-a = -(-4) = 4$ and $-(-a) = a = -4$.



Practice Set B

Find the opposite of each real number.

Exercise:

Problem: 8

Solution:

-8

Exercise:

Problem: 17

Solution:

-17

Exercise:

Problem: -6

Solution:

6

Exercise:

Problem: -15

Solution:

15

Exercise:

Problem: $-(-1)$

Solution:

-1, since $-(-1) = 1$

Exercise:

Problem: $-[-(-7)]$

Solution:

7

Exercise:

Problem:

Suppose that a is a positive number. What type of number is $-a$?

Solution:

If a is positive, $-a$ is negative.

Exercise:

Problem:

Suppose that a is a negative number. What type of number is $-a$?

Solution:

If a is negative, $-a$ is positive.

Exercise:

Problem:

Suppose we do not know the sign of the number m. Can we say that $-m$ is positive, negative, or that we do notknow ?

Solution:

We must say that we do not know.

Exercises

Exercise:

Problem:

A number is denoted as positive if it is directly preceded by ______________________ .

Solution:

a plus sign or no sign at all

Exercise:

Problem:

A number is denoted as negative if it is directly preceded by ______________________ .

For the following problems, how should the real numbers be read ? (Write in words.)

Exercise:

Problem: -5

Solution:

a negative five

Exercise:

Problem: -3

Exercise:

Problem: 12

Solution:

twelve

Exercise:

Problem: 10

Exercise:

Problem: $-(-4)$

Solution:

negative negative four

Exercise:

Problem: $-(-1)$

For the following problems, write the expressions in words.

Exercise:

Problem: $5+7$

Solution:

five plus seven

Exercise:

Problem: $2+6$

Exercise:

Problem: $11 + (-2)$

Solution:

eleven plus negative two

Exercise:

Problem: $1 + (-5)$

Exercise:

Problem: $6 - (-8)$

Solution:

six minus negative eight

Exercise:

Problem: $0 - (-15)$

Rewrite the following problems in a simpler form.

Exercise:

Problem: $-(-8)$

Solution:

$-(-8) = 8$

Exercise:

Problem: $-(-5)$

Exercise:

Problem: $-(-2)$

Solution:

2

Exercise:

Problem: $-(-9)$

Exercise:

Problem: $-(-1)$

Solution:

1

Exercise:

Problem: $-(-4)$

Exercise:

Problem: $-[-(-3)]$

Solution:

-3

Exercise:

Problem: $-[-(-10)]$

Exercise:

Problem: $-[-(-6)]$

Solution:

-6

Exercise:

Problem: $-[-(-15)]$

Exercise:

Problem: $-\{-[-(-26)]\}$

Solution:

26

Exercise:

Problem: $-\{-[-(-11)]\}$

Exercise:

Problem: $-\{-[-(-31)]\}$

Solution:

31

Exercise:

Problem: $-\{-[-(-14)]\}$

Exercise:

Problem: $-[-(12)]$

Solution:

12

Exercise:

Problem: $-[-(2)]$

Exercise:

Problem: $-[-(17)]$

Solution:

17

Exercise:

Problem: $-[-(42)]$

Exercise:

Problem: $5-(-2)$

Solution:

$5-(-2)=5+2=7$

Exercise:

Problem: $6-(-14)$

Exercise:

Problem: $10-(-6)$

Solution:

16

Exercise:

Problem: $18-(-12)$

Exercise:

Problem: $31-(-1)$

Solution:

32

Exercise:

Problem: $54-(-18)$

Exercise:

Problem: $6-(-3)-(-4)$

Solution:

13

Exercise:

Problem: $2-(-1)-(-8)$

Exercise:

Problem: $15-(-6)-(-5)$

Solution:

26

Exercise:

Problem: $24-(-8)-(-13)$

Exercises for Review

Exercise:

Problem:

(**[link]**) There is only one real number for which $(5a)^2 = 5a^2$. What is the number?

Solution:

0

Exercise:

Problem: (**[link]**) Simplify $(3xy)(2x^2y^3)(4x^2y^4)$.

Exercise:

Problem: (**[link]**) Simplify $x^{n+3} \cdot x^5$.

Solution:

x^{n+8}

Exercise:

Problem: (**[link]**) Simplify $(a^3b^2c^4)^4$.

Exercise:

Problem: (**[link]**) Simplify $\left(\frac{4a^2b}{3xy^3}\right)^2$.

Solution:

$\frac{16a^4b^2}{9x^2y^6}$

Signed Numbers: Absolute Value
This module is from Fundamentals of Mathematics by Denny Burzynski and Wade Ellis, Jr. This module discusses absolute value. By the end of the module students should understand the geometric and algebraic definitions of absolute value.

Section Overview

- Geometric Definition of Absolute Value
- Algebraic Definition of Absolute Value

Geometric Definition of Absolute Value

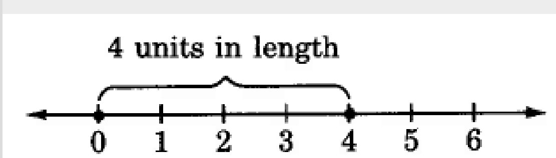
Absolute Value-Geometric Approach
Geometric definition of absolute value:
The **absolute value** of a number a, denoted $\mid a \mid$, is the distance from a to 0 on the number line.

Absolute value answers the question of "how far," and not "which way." The phrase "how far" implies "length" and *length is always a nonnegative quantity*. Thus, the absolute value of a number is a nonnegative number.

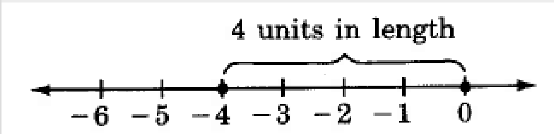
Sample Set A

Determine each value.

Example:
$\mid 4 \mid = 4$



Example:
$|-4|=4$



Example:
$|0|=0$

Example:
$-|5|=-5$. The quantity on the left side of the equal sign is read as "negative the absolute value of 5." The absolute value of 5 is 5. Hence, negative the absolute value of 5 is -5.

Example:
$-|-3|=-3$. The quantity on the left side of the equal sign is read as "negative the absolute value of -3." The absolute value of -3 is 3. Hence, negative the absolute value of -3 is $-(3)=-3$.

Practice Set A

By reasoning geometrically, determine each absolute value.

Exercise:

Problem: $|7|$

Solution:

7

Exercise:

Problem: $\mid -3 \mid$

Solution:

3

Exercise:

Problem: $\mid 12 \mid$

Solution:

12

Exercise:

Problem: $\mid 0 \mid$

Solution:

0

Exercise:

Problem: $- \mid 9 \mid$

Solution:

-9

Exercise:

Problem: $- \mid -6 \mid$

Solution:

-6

Algebraic Definition of Absolute Value

From the problems in [link], we can suggest the following algebraic definition of absolute value. Note that the definition has two parts.

Absolute Value—Algebraic Approach
Algebraic definition of absolute value
The absolute value of a number a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } < 0 \end{cases}$$

The algebraic definition takes into account the fact that the number a could be either positive or zero $(a \geq 0)$ or negative $(a < 0)$.

1. If the number a is positive or zero $(a \geq 0)$, the upper part of the definition applies. The upper part of the definition tells us that if the number enclosed in the absolute value bars is a nonnegative number, the absolute value of the number is the number itself.
2. The lower part of the definition tells us that if the number enclosed within the absolute value bars is a negative number, the absolute value of the number is the opposite of the number. The opposite of a negative number is a positive number.

Note: The definition says that the vertical absolute value lines may be eliminated only if we know whether the number inside is positive or negative.

Sample Set B

Use the algebraic definition of absolute value to find the following values.

Example:
$\mid 8 \mid$. The number enclosed within the absolute value bars is a nonnegative number, so the upper part of the definition applies. This part says that the absolute value of 8 is 8 itself.
$\mid 8 \mid = 8$

Example:
$\mid -3 \mid$. The number enclosed within absolute value bars is a negative number, so the lower part of the definition applies. This part says that the absolute value of -3 is the opposite of -3, which is $-(-3)$. By the definition of absolute value and the double-negative property,
$\mid -3 \mid = -(-3) = 3$

Practice Set B

Use the algebraic definition of absolute value to find the following values.

Exercise:

Problem: $\mid 7 \mid$

Solution:

7

Exercise:

Problem: $\mid 9 \mid$

Solution:

9

Exercise:

Problem: $\mid -12 \mid$

Solution:

12

Exercise:

Problem: $\mid -5 \mid$

Solution:

5

Exercise:

Problem: $- \mid 8 \mid$

Solution:

-8

Exercise:

Problem: $- \mid 1 \mid$

Solution:

-1

Exercise:

Problem: $- \mid -52 \mid$

Solution:

-52

Exercise:

Problem: $-\mid -31 \mid$

Solution:

-31

Exercises

Determine each of the values.

Exercise:

Problem: $\mid 5 \mid$

Solution:

5

Exercise:

Problem: $\mid 3 \mid$

Exercise:

Problem: $\mid 6 \mid$

Solution:

6

Exercise:

Problem: $\mid -9 \mid$

Exercise:

Problem: $\mid -1 \mid$

Solution:

1

Exercise:

Problem: $\mid -4 \mid$

Exercise:

Problem: $- \mid 3 \mid$

Solution:

-3

Exercise:

Problem: $- \mid 7 \mid$

Exercise:

Problem: $- \mid -14 \mid$

Solution:

-14

Exercise:

Problem: $\mid 0 \mid$

Exercise:

Problem: $\mid -26 \mid$

Solution:

26

Exercise:

Problem: $-\mid -26\mid$

Exercise:

Problem: $-(-\mid 4\mid)$

Solution:

4

Exercise:

Problem: $-(-\mid 2\mid)$

Exercise:

Problem: $-(-\mid -6\mid)$

Solution:

6

Exercise:

Problem: $-(-\mid -42\mid)$

Exercise:

Problem: $\mid 5\mid - \mid -2\mid$

Solution:

3

Exercise:

Problem: $\mid -2 \mid^3$

Exercise:

Problem: $\mid -(2 \cdot 3) \mid$

Solution:

6

Exercise:

Problem: $\mid -2 \mid - \mid -9 \mid$

Exercise:

Problem: $(\mid -6 \mid + \mid 4 \mid)^2$

Solution:

100

Exercise:

Problem: $(\mid -1 \mid - \mid 1 \mid)^3$

Exercise:

Problem: $(\mid 4 \mid + \mid -6 \mid)^2 - (\mid -2 \mid)^3$

Solution:

92

Exercise:

Problem: $-[|-10|-6]^2$

Exercise:

Problem: $-\left\{-[-\mid -4 \mid + \mid -3 \mid]^3\right\}^2$

Solution:

-1

Exercise:

Problem:

A Mission Control Officer at Cape Canaveral makes the statement "lift-off, *T* minus 50 seconds." How long is it before lift-off?

Exercise:

Problem:

Due to a slowdown in the industry, a Silicon Valley computer company finds itself in debt \$2,400,000. Use absolute value notation to describe this company's debt.

Solution:

$-\$ \mid -2{,}400{,}000 \mid$

Exercise:

Problem:

A particular machine is set correctly if upon action its meter reads 0. One particular machine has a meter reading of -1.6 upon action. How far is this machine off its correct setting?

Exercises for Review

Exercise:

Problem: ([link]) Find the sum: $\frac{9}{70} + \frac{5}{21} + \frac{8}{15}$.

Solution:

$\frac{9}{10}$

Exercise:

Problem: ([link]) Find the value of $\frac{\frac{3}{10} + \frac{4}{12}}{\frac{19}{20}}$.

Exercise:

Problem: ([link]) Convert $3.2\frac{3}{5}$ to a fraction.

Solution:

$3\frac{13}{50} \text{or} \frac{163}{50}$

Exercise:

Problem:

([link]) The ratio of acid to water in a solution is $\frac{3}{8}$. How many mL of acid are there in a solution that contain 112 mL of water?

Exercise:

Problem: ([link]) Find the value of $-6 - (-8)$.

Solution:

2

Basic Properties of Real Numbers: The Real Number Line and the Real Numbers
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The symbols, notations, and properties of numbers that form the basis of algebra, as well as exponents and the rules of exponents, are introduced in this chapter. Each property of real numbers and the rules of exponents are expressed both symbolically and literally. Literal explanations are included because symbolic explanations alone may be difficult for a student to interpret. Objectives of this module: be familiar with the real number line and the real numbers, understand the ordering of the real numbers.

Overview

- The Real Number Line
- The Real Numbers
- Ordering the Real Numbers

The Real Number Line

Real Number Line
In our study of algebra, we will use several collections of numbers. The **real number line** allows us to **visually** display the numbers in which we are interested.

A line is composed of infinitely many points. To each point we can associate a unique number, and with each number we can associate a particular point.

Coordinate
The number associated with a point on the number line is called the **coordinate** of the point.

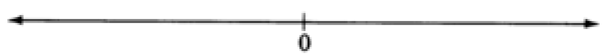
Graph
The point on a line that is associated with a particular number is called the **graph** of that number.

We construct the real number line as follows:

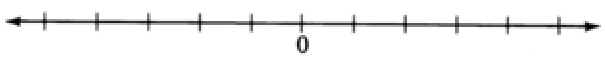
Construction of the Real Number Line

1. Draw a horizontal line.

2. Choose any point on the line and label it 0. This point is called the **origin**.



3. Choose a convenient length. This length is called "1 unit." Starting at 0, mark this length off in both directions, being careful to have the lengths look like they are about the same.



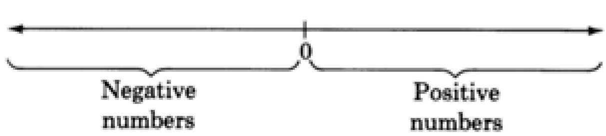
We now define a real number.

Real Number

A **real number** is any number that is the coordinate of a point on the real number line.

Positive and Negative Real Numbers

The collection of these infinitely many numbers is called the **collection of real numbers**. The real numbers whose graphs are to the right of 0 are called the **positive real numbers**. The real numbers whose graphs appear to the left of 0 are called the **negative real numbers**.

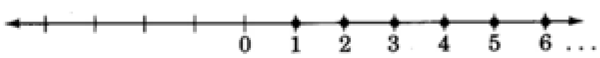


The number 0 is neither positive nor negative.

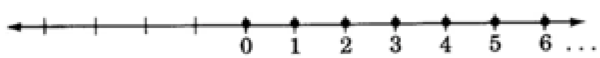
The Real Numbers

The collection of real numbers has many subcollections. The subcollections that are of most interest to us are listed below along with their notations and graphs.

Natural Numbers
The **natural numbers** (N)**:** $\{1,\ 2, 3,\ \ldots\}$

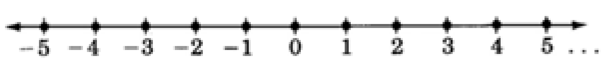


Whole Numbers
The **whole numbers** (W)**:** $\{0,\ 1,\ 2, 3,\ \ldots\}$



Notice that every natural number is a whole number.

Integers
The **integers** (Z)**:** $\{\ldots,\ -3, -2, -1, 0, 1,\ 2,\ 3,\ \ldots\}$



Notice that every whole number is an integer.

Rational Numbers
The **rational numbers** (Q)**:** Rational numbers are real numbers that can be written in the form a/b, where a and b are integers, and $b \neq 0$.

Fractions
Rational numbers are commonly called **fractions.**

Division by 1
Since b can equal 1, every integer is a rational number: $\frac{a}{1} = a$.

Division by 0
Recall that $10/2 = 5$ since $2 \cdot 5 = 10$. However, if $10/0 = x$, then $0 \cdot x = 10$. But $0 \cdot x = 0$, not 10. This suggests that no quotient exists.

Now consider $0/0 = x$. If $0/0 = x$, then $0 \cdot x = 0$. But this means that x could be any number, that is, $0/0 = 4$ since $0 \cdot 4 = 0$, or $0/0 = 28$ since $0 \cdot 28 = 0$. This suggests that the quotient is indeterminant.

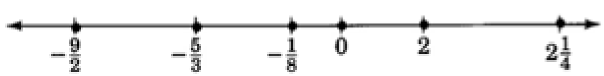
$x/0$ Is Undefined or Indeterminant
Division by 0 is undefined or indeterminant.

Do not divide by 0.

Rational numbers have decimal representations that either terminate or do not terminate but contain a repeating block of digits. Some examples are:

$$\underset{\text{Terminating}}{\frac{3}{4} = 0.75} \qquad \underset{\text{Nonterminating, but repeating}}{\frac{15}{11} = 1.36363636\ldots}$$

Some rational numbers are graphed below.



Irrational Numbers
The **irrational numbers** (Ir)**:** Irrational numbers are numbers that cannot be written as the quotient of two integers. They are numbers whose decimal representations are nonterminating and nonrepeating. Some examples are

$$4.01001000100001\ldots \qquad \pi = 3.1415927\ldots$$

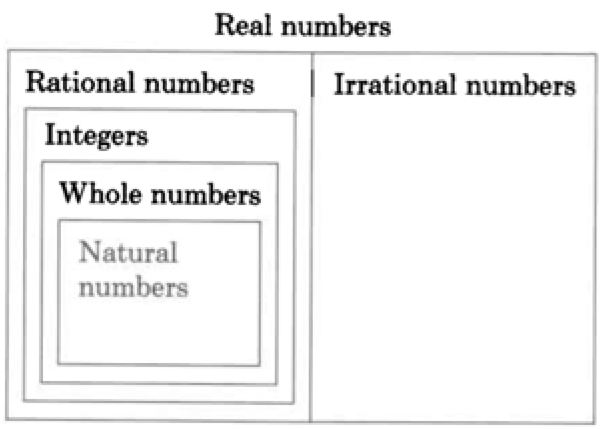
Notice that the collections of rational numbers and irrational numbers have no numbers in common.

When graphed on the number line, the rational and irrational numbers account for every point on the number line. Thus each point on the number line has a coordinate that is either a rational or an irrational number.

In summary, we have

Sample Set A

The summaray chart illustrates that



Example:
Every natural number is a real number.

Example:
Every whole number is a real number.

Example:
No integer is an irrational number.

Practice Set A

Exercise:

Problem: Is every natural number a whole number?

Solution:

yes

Exercise:

Problem: Is every whole number an integer?

Solution:

yes

Exercise:

Problem: Is every integer a rational number?

Solution:

yes

Exercise:

Problem: Is every rational number a real number?

Solution:

yes

Exercise:

Problem: Is every integer a natural number?

Solution:

no

Exercise:

Problem: Is there an integer that is a natural number?

Solution:

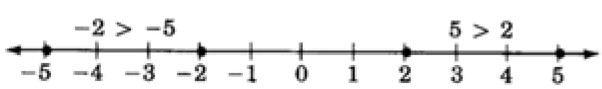
yes

Ordering the Real Numbers

Ordering the Real Numbers
A real number b is said to be greater than a real number a, denoted $b > a$, if the graph of b is to the right of the graph of a on the number line.

Sample Set B

As we would expect, $5 > 2$ since 5 is to the right of 2 on the number line. Also, $-2 > -5$ since -2 is to the right of -5 on the number line.



Practice Set B

Exercise:

Problem: Are all positive numbers greater than 0?

Solution:

yes

Exercise:

Problem: Are all positive numbers greater than all negative numbers?

Solution:

yes

Exercise:

Problem: Is 0 greater than all negative numbers?

Solution:

yes

Exercise:

Problem:

Is there a largest positive number? Is there a smallest negative number?

Solution:

no, no

Exercise:

Problem:

How many real numbers are there? How many real numbers are there between 0 and 1?

Solution:

infinitely many, infinitely many

Sample Set C

Example:
What integers can replace x so that the following statement is true?

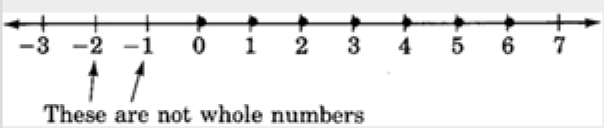
$$-4 \leq x < 2$$

This statement indicates that the number represented by x is between -4 and 2. Specifically, -4 is less than or equal to x, and at the same time, x is strictly less than 2. This statement is an example of a compound inequality.

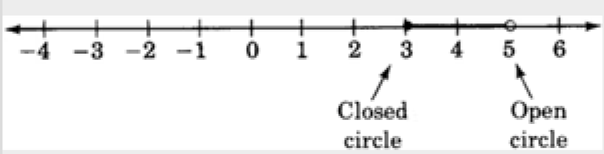
The integers are $-4, \ -3, \ -2, \ -1, \ 0, \ 1$.

Example:
Draw a number line that extends from -3 to 7. Place points at all whole numbers between and including -2 and 6.

Example:
Draw a number line that extends from -4 to 6 and place points at all real numbers greater than or equal to 3 but strictly less than 5.



It is customary to use a **closed circle** to indicate that a point is included in the graph and an **open circle** to indicate that a point is not included.



Practice Set C

Exercise:

Problem:

What whole numbers can replace x so that the following statement is true?

$-3 \le x < 3$

Solution:

0, 1, 2

Exercise:

Problem:

Draw a number line that extends from -5 to 3 and place points at all numbers greater than or equal to -4 but strictly less than 2.

Solution:

Exercises

For the following problems, next to each real number, note all collections to which it belongs by writing N for natural numbers, W for whole numbers, Z for integers, Q for rational numbers, Ir for irrational numbers, and R for real numbers. Some numbers may require more than one letter.

Exercise:

Problem: $\frac{1}{2}$

Solution:

Q, R

Exercise:

Problem: -12

Exercise:

Problem: 0

Solution:

W, Z, Q, R

Exercise:

Problem: $-24\frac{7}{8}$

Exercise:

Problem: $86.3333\ldots$

Solution:

Q, R

Exercise:

Problem: $49.125125125\ldots$

Exercise:

Problem: -15.07

Solution:

$Q,\ R$

For the following problems, draw a number line that extends from -3 to 3. Locate each real number on the number line by placing a point (closed circle) at its approximate location.

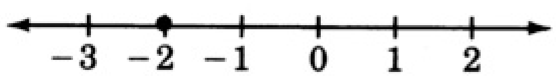
Exercise:

Problem: $1\frac{1}{2}$

Exercise:

Problem: -2

Solution:



Exercise:

Problem: $-\frac{1}{8}$

Exercise:

Problem: Is 0 a positive number, negative number, neither, or both?

Solution:

neither

Exercise:

Problem:

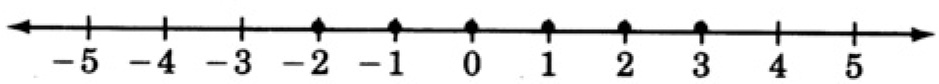
An integer is an even integer if it can be divided by 2 without a remainder; otherwise the number is odd. Draw a number line that extends from -5 to 5 and place points at all negative even integers and at all positive odd integers.

Exercise:

Problem:

Draw a number line that extends from -5 to 5. Place points at all integers strictly greater than -3 but strictly less than 4.

Solution:



For the following problems, draw a number line that extends from -5 to 5. Place points at all real numbers between and including each pair of numbers.

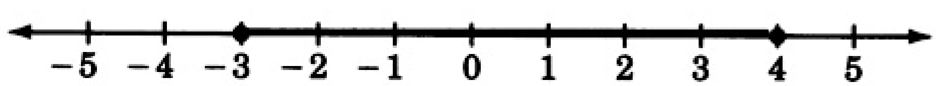
Exercise:

Problem: -5 and -2

Exercise:

Problem: -3 and 4

Solution:



Exercise:

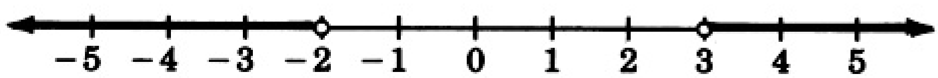
Problem: -4 and 0

Exercise:

Problem:

Draw a number line that extends from -5 to 5. Is it possible to locate any numbers that are strictly greater than 3 but also strictly less than -2?

Solution:



; no

For the pairs of real numbers shown in the following problems, write the appropriate relation symbol $(<,\ >,\ =)$ in place of the $*$.

Exercise:

Problem: $-5 * -1$

Exercise:

Problem: $-3 * 0$

Solution:

$<$

Exercise:

Problem: $-4 * 7$

Exercise:

Problem: $6 * -1$

Solution:

$>$

Exercise:

Problem: $-\frac{1}{4} * -\frac{3}{4}$

Exercise:

Problem: Is there a largest real number? If so, what is it?

Solution:

no

Exercise:

Problem: Is there a largest integer? If so, what is it?

Exercise:

Problem: Is there a largest two-digit integer? If so, what is it?

Solution:

99

Exercise:

Problem: Is there a smallest integer? If so, what is it?

Exercise:

Problem: Is there a smallest whole number? If so, what is it?

Solution:

yes, 0

For the following problems, what numbers can replace x so that the following statements are true?

Exercise:

Problem: $-1 \leq x \leq 5 \quad x \text{ an integer}$

Exercise:

Problem: $-7 < x < -1, \quad x \text{ an integer}$

Solution:

$-6, \ -5, \ -4, \ -3, \ -2$

Exercise:

Problem: $-3 \leq x \leq 2, \quad x \text{ a natural number}$

Exercise:

Problem: $-15 < x \leq -1, \quad x \text{ a natural number}$

Solution:

There are no natural numbers between −15 and −1.

Exercise:

Problem: $-5 \leq x < 5, \quad x \text{ a whole number}$

Exercise:

Problem:

The temperature in the desert today was ninety-five degrees. Represent this temperature by a rational number.

Solution:

$\left(\frac{95}{1}\right)^{\circ}$

Exercise:

Problem:

The temperature today in Colorado Springs was eight degrees below zero. Represent this temperature with a real number.

Exercise:

Problem: Is every integer a rational number?

Solution:

Yes, every integer is a rational number.

Exercise:

Problem: Is every rational number an integer?

Exercise:

Problem:

Can two rational numbers be added together to yield an integer? If so, give an example.

Solution:

Yes. $\frac{1}{2}+\frac{1}{2}=1 \text{or} 1+1=2$

For the following problems, on the number line, how many units (intervals) are there between?

Exercise:

Problem: 0 and 2?

Exercise:

Problem: -5 and 0?

Solution:

5 units

Exercise:

Problem: 0 and 6?

Exercise:

Problem: -8 and 0?

Solution:

8 units

Exercise:

Problem: -3 and 4?

Exercise:

Problem: m and n, $m > n$?

Solution:

$m - n$ units

Exercise:

Problem: $-a$ and $-b$, $-b > -a$?

Exercises for Review

Exercise:

Problem: (**[link]**) Find the value of $6+3(15-8)-4$.

Solution:

23

Exercise:

Problem: (**[link]**) Find the value of $5(8-6)+3(5+2 \cdot 3)$.

Exercise:

Problem:

(**[link]**) Are the statements $y < 4$ and $y \geq 4$ the same or different?

Solution:

different

Exercise:

Problem:

(**[link]**) Use algebraic notation to write the statement "six times a number is less than or equal to eleven."

Exercise:

Problem:

(**[link]**) Is the statement $8(15-3 \cdot 4)-3 \cdot 7 \geq 3$ true or false?

Solution:

true

Basic Properties of Real Numbers: Symbols and Notations
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The symbols, notations, and properties of numbers that form the basis of algebra, as well as exponents and the rules of exponents, are introduced in this chapter. Each property of real numbers and the rules of exponents are expressed both symbolically and literally. Literal explanations are included because symbolic explanations alone may be difficult for a student to interpret. Topics covered in this module: understand the difference between variables and constants, be familiar with the symbols of operation, equality, and inequality, be familiar with grouping symbols, be able to correctly use the order of operations.

Overview

- Variables and Constants
- Symbols of Operation, Equality, and Inequality
- Grouping Symbols
- The Order of Operations

Variables and Constants

A basic characteristic of algebra is the use of symbols (usually letters) to represent numbers.

Variable
A letter or symbol that represents any member of a collection of two or more numbers is called a **variable**.

Constant
A letter or symbol that represents a specific number, known or unknown is called a **constant**.

In the following examples, the letter x is a variable since it can be any member of the collection of numbers $\{35, 25, 10\}$. The letter h is a constant since it can assume only the value 5890.

Example:
Suppose that the streets on your way from home to school have speed limits of 35 mph, 25 mph, and 10 mph. In algebra we can let the letter x represent our speed

as we travel from home to school. The maximum value of x depends on what section of street we are on. The letter x can assume any one of the **various** values 35,25,10.

Example:
Suppose that in writing a term paper for a geography class we need to specify the height of Mount Kilimanjaro. If we do not happen to know the height of the mountain, we can represent it (at least temporarily) on our paper with the letter h. Later, we look up the height in a reference book and find it to be 5890 meters. The letter h can assume only the one value, 5890, and no others. The value of h is **constant**.

Symbols of Operation, Equality, and Inequality

Binary Operation

A **binary operation** on a collection of numbers is a process that assigns a number to two given numbers in the collection. The binary operations used in algebra are addition, subtraction, multiplication, and division.

Symbols of Operation

If we let x and y each represent a number, we have the following notations:

Addition	$x+y$			
Subtraction	$x-y$			
Multiplication	$x \cdot y$	$(x)(y)$	$x(y)$	xy
Division	$\frac{x}{y}$	x/y	$x \div y$	$y\sqrt{x}$

Sample Set A

Example:
$a+b$ represents the **sum** of a and b.

Example:

$4 + y$ represents the **sum** of 4 and y.

Example:
$8 - x$ represents the **difference** of 8 and x.

Example:
$6x$ represents the **product** of 6 and x.

Example:
ab represents the **product** of a and b.

Example:
$h3$ represents the **product** of h and 3.

Example:
$(14.2)a$ represents the **product** of 14.2 and a.

Example:
$(8)\ (24)$ represents the **product** of 8 and 24.

Example:
$5 \cdot 6(b)$ represents the **product** of 5,6, and b.

Example:
$\frac{6}{x}$ represents the **quotient** of 6 and x.

Practice Set A

Exercise:

Problem: Represent the product of 29 and x five different ways.

Solution:

$29 \cdot x, \quad 29x, \quad (29)(x), \quad 29(x), \quad (29)x$

If we let a and b represent two numbers, then a and b are related in exactly one of three ways:

Equality and Inequality Symbols

$a = b \quad a \text{ and } b \text{ are equal}$

$a > b \quad a \text{ is strictly greater than } b$

$a < b \quad a \text{ is strictly less than } b$

Some variations of these symbols include

$a \neq b \quad a \text{ is not equal to } b$

$a \geq b \quad a \text{ is greater than or equal to } b$

$a \leq b \quad a \text{ is less than or equal to } b$

The last five of the above symbols are inequality symbols. We can **negate** (change to the opposite) any of the above statements by drawing a line through the relation symbol (as in $a \neq b$), as shown below:

a is not greater than b can be expressed as either

$a \not> b$ or $a \leq$ b.

a is not less b than can be expressed as either

$a \not< b$ or $a \geq$ b.

$a < b$ and $a \not\geq b$ both indicate that a is less than b.

Grouping Symbols

Grouping symbols are used to indicate that a particular collection of numbers and meaningful operations are to be grouped together and considered as one number. The grouping symbols commonly used in algebra are

Parentheses:	$(\quad)$
Brackets:	$[\quad]$
Braces:	$\{\quad\}$
Bar:	

In a computation in which more than one operation is involved, grouping symbols help tell us which operations to perform first. If possible, we perform operations inside grouping symbols first.

Sample Set B

Example:
$(4+17)-6=21-6=15$

Example:
$8(3+6)=8(9)=72$

Example:
$5[8+(10-4)]=5[8+6]=5[14]=70$

Example:
$2\{3[4(17-11)]\}=2\{3[4(6)]\}=2\{3[24]\}=2\{72\}=144$

Example:

$\frac{9(5+1)}{24+3}$.

The fraction bar separates the two groups of numbers $9(5+1)$ and $24+3$. Perform the operations in the numerator and denominator separately.

$\frac{9(5+1)}{24+3} = \frac{9(6)}{24+3} = \frac{54}{24+3} = \frac{54}{27} = 2$

Practice Set B

Use the grouping symbols to help perform the following operations.

Exercise:

Problem: $3(1+8)$

Solution:

27

Exercise:

Problem: $4[2(11-5)]$

Solution:

48

Exercise:

Problem: $6\{2[2(10-9)]\}$

Solution:

24

Exercise:

Problem: $\frac{1+19}{2+3}$

Solution:

The following examples show how to use algebraic notation to write each expression.

Example:
9 minus y becomes $9 - y$

Example:
46 times x becomes 46x

Example:
7 times $(x + y)$ becomes $7(x + y)$

Example:
4 divided by 3, times z becomes $\left(\frac{4}{3}\right) z$

Example:
$(a - b)$ times $(b - a)$ divided by (2 times a) becomes $\frac{(a-b)\,(b-a)}{2a}$

Example:
Introduce a variable (**any** letter will do but here we'll let x represent the number) and use appropriate algebraic symbols to write the statement: A number plus 4 is strictly greater than 6. The answer is $x + 4 > 6$.

The Order of Operations

Suppose we wish to find the value of $16 + 4 \cdot 9$. We could

1. add 16 and 4, then multiply this sum by 9.
   $16 + 4 \cdot 9 = 20 \cdot 9 = 180$
2. multiply 4 and 9, then add 16 to this product.
   $16 + 4 \cdot 9 = 16 + 36 = 52$

We now have two values for one number. To determine the correct value we must use the standard **order of operations**.
Order of Operations

1. Perform all operations inside grouping symbols, beginning with the innermost set.
2. Perform all multiplications and divisions, as you come to them, moving left-to-right.
3. Perform all additions and subtractions, as you come to them, moving left-to-right.

As we proceed in our study of algebra, we will come upon another operation, exponentiation, that will need to be inserted before multiplication and division. (See Section [link].)

Sample Set C

USe the order of operations to find the value of each number.

Example:
$$\begin{array}{ll} 16 + 4 \cdot 9 & \text{Multiply first.} \\ = 16 + 36 & \text{Now add.} \\ = 52 & \end{array}$$

Example:
$$\begin{array}{ll} (27 - 8) + 7(6 + 12) & \text{Combine within parentheses.} \\ = 19 + 7(18) & \text{Multiply.} \\ = 19 + 126 & \text{Now add.} \\ = 145 & \end{array}$$

Example:

$8+2[4+3(6-1)]$ Begin with the innermost set of grouping symbols, $(\quad)$.

$=8+2[4+3(5)]$ Now work within the next set of grouping symbols, $[\quad]$.

$=8+2[4+15]$

$=8+2[19]$

$=8+38$

$=46$

Example:

$$\begin{aligned}\frac{6+4[2+3(19-17)]}{18-2[2(3)+2]} &= \frac{6+4[2+3(2)]}{18-2[6+2]}\\ &= \frac{6+4[2+6]}{18-2[8]}\\ &= \frac{6+4[8]}{18-16}\\ &= \frac{6+32}{2}\\ &= \frac{38}{2}\\ &= 19\end{aligned}$$

Practice Set C

Use the order of operations to find each value.

Exercise:

Problem: $25+8(3)$

Solution:

49

Exercise:

Problem: $2 + 3(18 - 5 \cdot 2)$

Solution:

26

Exercise:

Problem: $4 + 3[2 + 3(1 + 8 \div 4)]$

Solution:

37

Exercise:

Problem: $\frac{19+2\{5+2[18+6(4+1)]\}}{5 \cdot 6 - 3(5) - 2}$

Solution:

17

Exercises

For the following problems, use the order of operations to find each value.

Exercise:

Problem: $2 + 3(6)$

Solution:

20

Exercise:

Problem: $18 - 7(8 - 3)$

Exercise:

Problem: $8 \cdot 4 \div 16 + 5$

Solution:

7

Exercise:

Problem: $(21 + 4) \div 5 \cdot 2$

Exercise:

Problem: $3(8 + 2) \div 6 + 3$

Solution:

8

Exercise:

Problem: $6(4 + 1) \div (16 \div 8) - 15$

Exercise:

Problem: $6(4 - 1) + 8(3 + 7) - 20$

Solution:

78

Exercise:

Problem: $(8)\,(5) + 2(14) + (1)\,(10)$

Exercise:

Problem: $61 - 22 + 4[3(10) + 11]$

Solution:

203

Exercise:

Problem: $\frac{(1+16-3)}{7}+5(12)$

Exercise:

Problem: $\frac{8(6+20)}{8}+\frac{3(6+16)}{22}$

Solution:

29

Exercise:

Problem: $18\div 2+55$

Exercise:

Problem: $21\div 7\div 3$

Solution:

1

Exercise:

Problem: $85\div 5\cdot 5-85$

Exercise:

Problem: $(300-25)\div(6-3)$

Solution:

$91\frac{2}{3}$

Exercise:

Problem: $4\cdot 3+8\cdot 28-(3+17)+11(6)$

Exercise:

Problem: $2\{(7+7)+6[4(8+2)]\}$

Solution:

508

Exercise:

Problem: $0+10(0)+15[4(3)+1]$

Exercise:

Problem: $6.1(2.2+1.8)$

Solution:

24.4

Exercise:

Problem: $\frac{5.9}{2}+0.6$

Exercise:

Problem: $(4+7)\,(8-3)$

Solution:

55

Exercise:

Problem: $(10+5)\,(10+5)-4(60-4)$

Exercise:

Problem: $\left(\frac{5}{12}-\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{2}{3}\right)$

Solution:

1

Exercise:

Problem: $4\left(\frac{3}{5}-\frac{8}{15}\right)+9\left(\frac{1}{3}+\frac{1}{4}\right)$

Exercise:

Problem: $\frac{0}{5}+\frac{0}{1}+0[2+4(0)]$

Solution:

0

Exercise:

Problem: $0\cdot 9+4\cdot 0\div 7+0[2(2-2)]$

For the following problems, state whether the given statements are the same or different.

Exercise:

Problem: $x \geq y \quad \text{and} \quad x > y$

Solution:

different

Exercise:

Problem:

$x < y \quad \text{and} \quad x \ngeq y$

Exercise:

Problem: $x = y \quad \text{and} \quad y = x$

Solution:

same

Exercise:

Problem: Represent the product of 3 and x five different ways.

Exercise:

Problem: Represent the sum of a and b two different ways.

Solution:

$a+b$, $b+a$

For the following problems, rewrite each phrase using algebraic notation.

Exercise:

Problem: Ten minus three

Exercise:

Problem: x plus sixteen

Solution:

$x+16$

Exercise:

Problem: 51 divided by a

Exercise:

Problem: 81 times x

Solution:

$81x$

Exercise:

Problem: 3 times $(x+y)$

Exercise:

Problem: $(x + b)$ times $(x + 7)$

Solution:

$(x + b)(x + 7)$

Exercise:

Problem: 3 times x times y

Exercise:

Problem: x divided by (7 times b)

Solution:

$\frac{x}{7b}$

Exercise:

Problem: $(a + b)$ divided by $(a + 4)$

For the following problems, introduce a variable (any letter will do) and use appropriate algebraic symbols to write the given statement.

Exercise:

Problem: A number minus eight equals seventeen.

Solution:

$x - 8 = 17$

Exercise:

Problem: Five times a number, minus one, equals zero.

Exercise:

Problem: A number divided by six is greater than or equal to forty-four.

Solution:

$\frac{x}{6} \geq 44$

Exercise:

Problem: Sixteen minus twice a number equals five.

Determine whether the statements for the following problems are true or false.

Exercise:

Problem: $6 - 4(4)\,(1) \leq 10$

Solution:

true

Exercise:

Problem: $5(4 + 2 \cdot 10) \geq 110$

Exercise:

Problem: $8 \cdot 6 - 48 \leq 0$

Solution:

true

Exercise:

Problem: $\frac{20+4.3}{16} < 5$

Exercise:

Problem: $2[6(1 + 4) - 8] > 3(11 + 6)$

Solution:

false

Exercise:

$$6[4+8+3(26-15)]$$

$\neq$

Problem: $3[7(10-4)]$

Exercise:

Problem:

The number of different ways 5 people can be arranged in a row is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. How many ways is this?

Solution:

120

Exercise:

Problem:

A box contains 10 computer chips. Three chips are to be chosen at random. The number of ways this can be done is

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

How many ways is this?

Exercise:

Problem:

The probability of obtaining four of a kind in a five-card poker hand is

$$\frac{13 \cdot 48}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \div (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$$

What is this probability?

Solution:

$0.00024,$ or $\frac{1}{4165}$

Exercise:

Problem:

Three people are on an elevator in a five story building. If each person randomly selects a floor on which to get off, the probability that at least two people get off on the same floor is

$$1 - \frac{5 \cdot 4 \cdot 3}{5 \cdot 5 \cdot 5}$$

What is this probability?

The Language of Algebra

By the end of this section, you will be able to:

- Use variables and algebraic symbols

Use Variables and Algebraic Symbols

Suppose this year Greg is 20 years old and Alex is 23. You know that Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right? In the language of algebra, we say that Greg's age and Alex's age are **variables** and the 3 is a **constant**. The ages change ("vary") but the 3 years between them always stays the same ("constant"). Since Greg's age and Alex's age will always differ by 3 years, 3 is the *constant*.

In algebra, we use letters of the alphabet to represent variables. So if we call Greg's age *g*, then we could use $g+3$ to represent Alex's age. See [link].

Greg's age	**Alex's age**
12	15
20	23
35	38
$\boldsymbol{g}$	$\boldsymbol{g}+3$

The letters used to represent these changing ages are called *variables*. The letters most commonly used for variables are *x*, *y*, *a*, *b*, and *c*.

Note:
Variable
A **variable** is a letter that represents a number whose value may change.

Note:
Constant
A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using.

There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We’ll list the symbols used to indicate these operations below ([link]). You’ll probably recognize some of them.

Operation	Notation	Say:	The result is…
Addition	$a+b$	*a* plus *b*	the sum of *a* and *b*
Subtraction	$a-b$	*a* minus *b*	the difference of *a* and *b*
Multiplication	$a\cdot b, ab, (a)\,(b), (a)b, a\,(b)$	*a* times *b*	the product of *a* and *b*
Division	$a\div b, a/b, \frac{a}{b}, b\overline{)a}$	*a* divided by *b*	the quotient of *a* and *b*, *a* is called the dividend, and *b* is called the divisor

We perform these operations on two numbers. When translating from symbolic form to English, or from English to symbolic form, pay attention to the words “of” and “and.”

- The *difference of* 9 and 2 means subtract 9 and 2, in other words, 9 minus 2, which we write symbolically as $9-2$.
- The *product of* 4 and 8 means multiply 4 and 8, in other words 4 times 8, which we write symbolically as $4\cdot 8$.

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does 3*xy* mean $3\ \times\ y$ (‘three times *y*’) or $3\cdot x\cdot y$ (three times *x* times *y*)? To make it clear, use · or parentheses for multiplication.

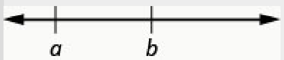
When two quantities have the same value, we say they are equal and connect them with an **equal sign**.

Note:
Equality Symbol
$a=b$ is read “*a* is equal to *b*”
The symbol “$=$” is called the **equal sign**.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols “<” and “>.”

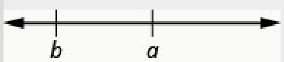
Note:
Inequality
Equation:

$$a < b \text{ is read “}a \text{ is less than } b\text{”}$$
$$a \text{ is to the left of } b \text{ on the number line}$$



Equation:

$$a > b \text{ is read “}a \text{ is greater than } b\text{”}$$
$$a \text{ is to the right of } b \text{ on the number line}$$



The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right ([link]). In general, $a < b$ is equivalent to $b > a$. For example $7 < 11$ is equivalent to $11 > 7$. And $a > b$ is equivalent to $b < a$. For example $17 > 4$ is equivalent to $4 < 17$.

Inequality Symbols	Words
$a \neq b$	a is *not equal to b*
$a < b$	a is *less than b*
$a \leq b$	a is *less than or equal to b*
$a > b$	a is *greater than b*
$a \geq b$	a is *greater than or equal to b*

Example:
Exercise:

Problem: Translate from algebra into English:

ⓐ $17 \leq 26$ ⓑ $8 \neq 17 - 3$ ⓒ $12 > 27 \div 3$ ⓓ $y + 7 < 19$

Solution:
Solution

ⓐ $17 \leq 26$
17 is less than or equal to 26

ⓑ $8 \neq 17 - 3$
8 is not equal to 17 minus 3

ⓒ $12 > 27 \div 3$
12 is greater than 27 divided by 3

ⓓ $y + 7 < 19$
y plus 7 is less than 19

Note:
Exercise:

Problem: Translate from algebra into English:

ⓐ $14 \leq 27$
ⓑ $19 - 2 \neq 8$
ⓒ $12 > 4 \div 2$
ⓓ $x - 7 < 1$

Solution:

ⓐ 14 is less than or equal to 27
ⓑ 19 minus 2 is not equal to 8
ⓒ 12 is greater than 4 divided by 2
ⓓ *x* minus 7 is less than 1

Note:
Exercise:

Problem: Translate from algebra into English:

ⓐ $19 \geq 15$
ⓑ $7 = 12 - 5$
ⓒ $15 \div 3 < 8$
ⓓ $y + 3 > 6$

Solution:

ⓐ 19 is greater than or equal to 15
ⓑ 7 is equal to 12 minus 5

ⓒ 15 divided by 3 is less than 8
ⓓ y plus 3 is greater than 6

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help to make clear which expressions are to be kept together and separate from other expressions. We will introduce three types now.

Note:
Grouping Symbols
Equation:

$$\begin{array}{ll} \text{Parentheses} & () \\ \text{Brackets} & [] \\ \text{Braces} & \{\} \end{array}$$

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.
Equation:

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24 \div \{13-2[1(6-5)+4]\}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb. In algebra, we have *expressions* and *equations*.

Note:
Expression
An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

An **expression** is like an English phrase. Here are some examples of expressions:

Expression	**Words**	**English Phrase**

Expression	Words	English Phrase
$3+5$	3 plus 5	the sum of three and five
$n-1$	*n* minus one	the difference of *n* and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	*x* divided by *y*	the quotient of *x* and *y*

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb.

An **equation** is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Note:
Equation
An **equation** is two expressions connected by an equal sign.

Here are some examples of equations.

Equation	English Sentence
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	*n* minus one equals fourteen.
$6 \cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	*x* is equal to fifty-three.
$y+9=2y-3$	*y* plus nine is equal to two *y* minus three.

Example:
Exercise:

Problem: Determine if each is an expression or an equation:

ⓐ $2(x+3)=10$
ⓑ $4(y-1)+1$
ⓒ $x \div 25$
ⓓ $y+8=40$

Solution:
Solution

ⓐ
$2(x+3)=10$ This is an *equation*—two expressions are connected with an equal sign.
ⓑ $4(y-1)+1$ This is an *expression*—no equal sign.
ⓒ $x \div 25$ This is an *expression*—no equal sign.
ⓓ
$y+8=40$ This is an *equation*—two expressions are connected with an equal sign.

Note:
Exercise:

Determine if each is an expression or an equation:
ⓐ $3(x-7)=27$
Problem: ⓑ $5(4y-2)-7$.

Solution:

ⓐ equation ⓑ expression

Note:
Exercise:

Determine if each is an expression or an equation:
ⓐ $y^3 \div 14$
Problem: ⓑ $4x-6=22$.

Solution:

ⓐ expression ⓑ equation

Key Concepts

- **Notation** **The result is…**

- $a+b$ — the sum of a and b
- $a-b$ — the difference of a and b
- $a \cdot b, ab, (a)(b)\ (a)b, a(b)$ — the product of a and b
- $a \div b, a/b, \frac{a}{b}, b\overline{)a}$ — the quotient of a and b

- **Inequality**
  - $a<b$ is read "a is less than b" — a is to the left of b on the number line
  - $a>b$ is read "a is greater than b" — a is to the right of b on the number line
- **Inequality Symbols** — **Words**
  - $a \neq b$ — a is **not equal to** b
  - $a<b$ — a is **less than** b
  - $a \leq b$ — a is **less than or equal to** b
  - $a>b$ — a is **greater than** b
  - $a \geq b$ — a is **greater than or equal to** b
- **Grouping Symbols**
  - Parentheses ( )
  - Brackets [ ]
  - Braces { }
- **Exponential Notation**
  - a^n means multiply a by itself, n times. The expression a^n is read a to the n^{th} power.

Practice Makes Perfect

Use Variables and Algebraic Symbols

In the following exercises, translate from algebra to English.

Exercise:

Problem: $16-9$

Solution:

16 minus 9, the difference of sixteen and nine

Exercise:

Problem: $3 \cdot 9$

Exercise:

Problem: $28 \div 4$

Solution:

28 divided by 4, the quotient of twenty-eight and four

Exercise:

Problem: $x + 11$

Exercise:

Problem: $(2)\,(7)$

Solution:

2 times 7, the product of two and seven

Exercise:

Problem: $(4)\,(8)$

Exercise:

Problem: $14 < 21$

Solution:

fourteen is less than twenty-one

Exercise:

Problem: $17 < 35$

Exercise:

Problem: $36 \geq 19$

Solution:

thirty-six is greater than or equal to nineteen

Exercise:

Problem: $6n = 36$

Exercise:

Problem: $y - 1 > 6$

Solution:

y minus 1 is greater than 6, the difference of *y* and one is greater than six

Exercise:

Problem: $y - 4 > 8$

Exercise:

Problem: $2 \leq 18 \div 6$

Solution:

2 is less than or equal to 18 divided by 6; 2 is less than or equal to the quotient of eighteen and six

Exercise:

Problem: $a \neq 1 \cdot 12$

In the following exercises, determine if each is an expression or an equation.

Exercise:

Problem: $9 \cdot 6 = 54$

Solution:

equation

Exercise:

Problem: $7 \cdot 9 = 63$

Exercise:

Problem: $5 \cdot 4 + 3$

Solution:

expression

Exercise:

Problem: $x + 7$

Exercise:

Problem: $x + 9$

Solution:

expression

Exercise:

Problem: $y - 5 = 25$

Everyday Math

Exercise:

Problem:

Car insurance Justin's car insurance has a \$750 deductible per incident. This means that he pays \$750 and his insurance company will pay all costs beyond \$750. If Justin files a claim for \$2,100.

ⓐ how much will he pay?
ⓑ how much will his insurance company pay?

Solution:

ⓐ \$750 ⓑ \$1,350

Exercise:

Problem:

Home insurance Armando's home insurance has a \$2,500 deductible per incident. This means that he pays \$2,500 and the insurance company will pay all costs beyond \$2,500. If Armando files a claim for \$19,400.

ⓐ how much will he pay?
ⓑ how much will the insurance company pay?

Glossary

coefficient
: The coefficient of a term is the constant that multiplies the variable in a term.

constant
: A constant is a number whose value always stays the same.

variable
: A variable is a letter that represents a number whose value may change.

term
: A term is a constant or the product of a constant and one or more variables.

equality symbol
: The symbol "$=$" is called the equal sign. We read $a=b$ as "a is equal to b."

equation
: An equation is two expressions connected by an equal sign.

Basic Operations with Real Numbers: Addition of Signed Numbers
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The basic operations with real numbers are presented in this chapter. The concept of absolute value is discussed both geometrically and symbolically. The geometric presentation offers a visual understanding of the meaning of |x|. The symbolic presentation includes a literal explanation of how to use the definition. Negative exponents are developed, using reciprocals and the rules of exponents the student has already learned. Scientific notation is also included, using unique and real-life examples. Objectives of this module: be able to add numbers with like signs and unlike signs, understand addition with zero.

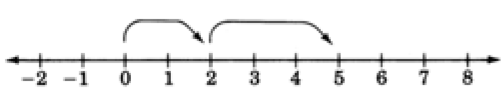
Overview

- Addition of Numbers with Like Signs
- Addition with Zero
- Addition of Numbers with Unlike Signs

Addition of Numbers with Like Signs

Let us add the two positive numbers 2 and 3. We perform this addition on the number line as follows.

We begin at 0, the origin.
Since 2 is positive, we move 2 units to the right.
Since 3 is positive, we move 3 more units to the right.
We are now located at 5.
Thus, $2 + 3 = 5$.



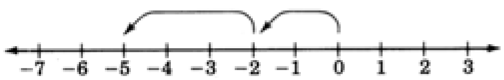
Summarizing, we have

$$(2 \text{ positive units}) + (3 \text{ positive units}) = (5 \text{ positive units})$$

Now let us add the two negative numbers -2 and -3. We perform this addition on the number line as follows.

We begin at 0, the origin.
Since -2 is negative, we move 2 units to the left.
Since -3 is negative, we move 3 more units to the left.
We are now located at -5.

Thus, $(-2) + (-3) = -5$.



Summarizing, we have

$$(2\,\text{negative units}) + (3\,\text{negative units}) = (5\,\text{negative units})$$

These two examples suggest that

$$(\text{positive number}) \ + \ (\text{positive number}) \ = \ (\text{positive number})$$
$$(\text{negative number}) \ + \ (\text{negative number}) \ = \ (\text{negative number})$$

Adding Numbers with the Same Sign
To add two real numbers that have the same sign, add the absolute values of the numbers and associate the common sign with the sum.

Sample Set A

Find the sums.

Example:
$3 + 7$

$$\text{Add these absolute values.}$$

$$\left.\begin{array}{l}|3| = 3 \\ |7| = 7\end{array}\right\} \quad 3 + 7 = 10$$

$$\text{The common sign is "+."}$$

$$3 + 7 = +10 \quad \text{or} \quad 3 + 7 = 10$$

Example:

$(-4) + (-9)$

$$\text{Add these absolute values.}$$

$$\left.\begin{array}{l}|-4| = 4 \\ |-9| = 9\end{array}\right\} \quad 4 + 9 = 13$$

$$\text{The common sign is "} - \text{."}$$

$(-4) + (-9) = -13$

Practice Set A

Find the sums.

Exercise:

Problem: $8 + 6$

Solution:

14

Exercise:

Problem: $41 + 11$

Solution:

52

Exercise:

Problem: $(-4)+(-8)$

Solution:

-12

Exercise:

Problem: $(-36)+(-9)$

Solution:

-45

Exercise:

Problem: $-14+(-20)$

Solution:

-34

Exercise:

Problem: $-\frac{2}{3}+\left(-\frac{5}{3}\right)$

Solution:

$-\frac{7}{3}$

Exercise:

Problem: $-2.8+(-4.6)$

Solution:

-7.4

Addition with Zero

Notice that

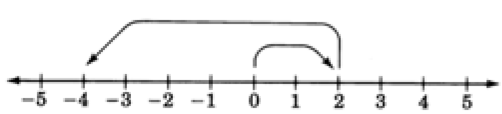
Addition with 0
$(0)\ +\ (\text{a positive number})\ =\ (\text{that same positive number})$
$(0)\ +\ (\text{a negative number})\ =\ (\text{that same negative number})$

The Additive Identity Is 0
Since adding 0 to a real number leaves that number unchanged, 0 is called the **additive identity**.

Addition of Numbers with Unlike Signs

Now let us perform the addition $2 + (-6)$. These two numbers have unlike signs. This type of addition can also be illustrated using the number line.

We begin at 0, the origin.
Since 2 is positive, we move 2 units to the right.
Since -6 is negative, we move, from the 2, 6 units to the left.
We are now located at -4.



A rule for adding two numbers that have unlike signs is suggested by noting that if the signs are disregarded, 4 can be obtained from 2 and 6 by **subtracting** 2 from 6. But 2 and 6 are precisely the absolute values of 2 and -6. Also, notice that the sign of the number with the larger absolute value is negative and that the sign of the resulting sum is negative.

Adding Numbers with Unlike Signs

To add two real numbers that have unlike signs, subtract the smaller absolute value from the larger absolute value and associate the sign of the number with the larger absolute value with this difference.

Sample Set B

Find the following sums.

Example:

$7+(-2)$

$|7|=7$ $\quad$ $|-2|=2$

Larger absolute value. Sign is "+". $\quad$ Smaller absolute value.

Subtract absolute values: $7-2=5$.

Attach the proper sign: "+".

$7+(-2)=+5$ or $7+(-2)=5$

Example:

$3+(-11)$

$|3|=3$ $\quad$ $|-11|=11$

Smaller absolute value. $\quad$ Larger absolute value. Sign is "−".

Subtract absolute values: $11-3=8$.

Attach the proper sign: "−".

$3+(-11)=-8$

Example:

The morning temperature on a winter's day in Lake Tahoe was -12 degrees. The afternoon temperature was 25 degrees warmer. What was the afternoon temperature?

We need to find $-12+25$.

$|-12| = 12$ $\quad$ $|25| = 25$

Smaller absolute value. $\quad$ Larger absolute value. Sign is "+".

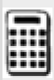
Subtract absolute values: $25 - 12 = 13$.

Attach the proper sign: "+".

$-12 + 25 = 13$

Thus, the afternoon temperature is 13 degrees.

Example:



Add $-147 + 84$. **Display Reads**

Type	147	147
Press	$\boxed{+/-}$	-147
Press	$\boxed{+}$	-147
Type	84	84
Press	$\boxed{=}$	-63

Practice Set B

Find the sums.

Exercise:

Problem: $4 + (-3)$

Solution:

1

Exercise:

Problem: $-3+5$

Solution:

2

Exercise:

Problem: $15+(-18)$

Solution:

-3

Exercise:

Problem: $0+(-6)$

Solution:

-6

Exercise:

Problem: $-26+12$

Solution:

-14

Exercise:

Problem: $35+(-78)$

Solution:

-43

Exercise:

Problem: $15 + (-10)$

Solution:

5

Exercise:

Problem: $1.5 + (-2)$

Solution:

-0.5

Exercise:

Problem: $-8 + 0$

Solution:

-8

Exercise:

Problem: $0 + (0.57)$

Solution:

0.57

Exercise:

Problem: $-879 + 454$

Solution:

-425

Exercise:

Problem: $-1345.6 + (-6648.1)$

Solution:

-7993.7

Exercises

Find the sums for the the following problems.

Exercise:

Problem: $4 + 12$

Solution:

16

Exercise:

Problem: $8 + 6$

Exercise:

Problem: $6 + 2$

Solution:

8

Exercise:

Problem: $7 + 9$

Exercise:

Problem: $(-3) + (-12)$

Solution:

-15

Exercise:

Problem: $(-6) + (-20)$

Exercise:

Problem: $(-4) + (-8)$

Solution:

-12

Exercise:

Problem: $(-11) + (-8)$

Exercise:

Problem: $(-16) + (-8)$

Solution:

-24

Exercise:

Problem: $(-2) + (-15)$

Exercise:

Problem: $14 + (-3)$

Solution:

11

Exercise:

Problem: $21 + (-4)$

Exercise:

Problem: $14 + (-6)$

Solution:

8

Exercise:

Problem: $18 + (-2)$

Exercise:

Problem: $10 + (-8)$

Solution:

2

Exercise:

Problem: $40 + (-31)$

Exercise:

Problem: $(-3) + (-12)$

Solution:

-15

Exercise:

Problem: $(-6) + (-20)$

Exercise:

Problem: $10 + (-2)$

Solution:

8

Exercise:

Problem: $8 + (-15)$

Exercise:

Problem: $-2 + (-6)$

Solution:

-8

Exercise:

Problem: $-11 + (-14)$

Exercise:

Problem: $-9 + (-6)$

Solution:

-15

Exercise:

Problem: $-1 + (-1)$

Exercise:

Problem: $-16 + (-9)$

Solution:

-25

Exercise:

Problem: $-22 + (-1)$

Exercise:

Problem: $0 + (-12)$

Solution:

-12

Exercise:

Problem: $0 + (-4)$

Exercise:

Problem: $0 + (24)$

Solution:

24

Exercise:

Problem: $-6 + 1 + (-7)$

Exercise:

Problem: $-5+(-12)+(-4)$

Solution:

-21

Exercise:

Problem: $-5+5$

Exercise:

Problem: $-7+7$

Solution:

0

Exercise:

Problem: $-14+14$

Exercise:

Problem: $4+(-4)$

Solution:

0

Exercise:

Problem: $9+(-9)$

Exercise:

Problem: $84+(-61)$

Solution:

23

Exercise:

Problem: $13 + (-56)$

Exercise:

Problem: $452 + (-124)$

Solution:

328

Exercise:

Problem: $636 + (-989)$

Exercise:

Problem: $1811 + (-935)$

Solution:

876

Exercise:

Problem: $-373 + (-14)$

Exercise:

Problem: $-1221 + (-44)$

Solution:

-1265

Exercise:

Problem: $-47.03 + (-22.71)$

Exercise:

Problem: $-1.998 + (-4.086)$

Solution:

-6.084

Exercise:

Problem: $[(-3) + (-4)] + [(-6) + (-1)]$

Exercise:

Problem: $[(-2) + (-8)] + [(-3) + (-7)]$

Solution:

-20

Exercise:

Problem: $[(-3) + (-8)] + [(-6) + (-12)]$

Exercise:

Problem: $[(-8) + (-6)] + [(-2) + (-1)]$

Solution:

-17

Exercise:

Problem: $[4+(-12)]+[12+(-3)]$

Exercise:

Problem: $[5+(-16)]+[4+(-11)]$

Solution:

-18

Exercise:

Problem: $[2+(-4)]+[17+(-19)]$

Exercise:

Problem: $[10+(-6)]+[12+(-2)]$

Solution:

14

Exercise:

Problem: $9+[(-4)+7]$

Exercise:

Problem: $14+[(-3)+5]$

Solution:

16

Exercise:

Problem: $[2+(-7)]+(-11)$

Exercise:

Problem: $[14 + (-8)] + (-2)$

Solution:

4

Exercise:

Problem:

In order for a small business to break even on a project, it must have sales of $\$21,000$. If the amount of sales was $\$15,000$, how much money did this company fall short?

Exercise:

Problem:

Suppose a person has $\$56.00$ in his checking account. He deposits $\$100.00$ into his checking account by using the automatic teller machine. He then writes a check for $\$84.50$. If an error causes the deposit not to be listed into this person's account, what is this person's checking balance?

Solution:

$-\$28.50$

Exercise:

Problem:

A person borrows $\$7.00$ on Monday and then $\$12.00$ on Tuesday. How much has this person borrowed?

Exercise:

Problem:

A person borrows \$11.00 on Monday and then pays back \$8.00 on Tuesday. How much does this person owe?

Solution:

\$3.00

Exercises for Review

Exercise:

Problem: ([link]) Determine the value of $|-8|$.

Solution:

8

Exercise:

Problem: ([link]) Determine the value of $(|2|+|4|^2)+|-5|^2$.

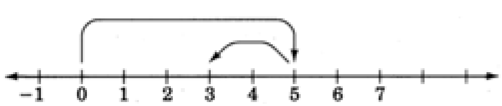
Basic Operations with Real Numbers: Subtraction of Signed Numbers
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The basic operations with real numbers are presented in this chapter. The concept of absolute value is discussed both geometrically and symbolically. The geometric presentation offers a visual understanding of the meaning of |x|. The symbolic presentation includes a literal explanation of how to use the definition. Negative exponents are developed, using reciprocals and the rules of exponents the student has already learned. Scientific notation is also included, using unique and real-life examples. Objectives of this module: understand the definition of subtraction, be able to subtract signed numbers.

Overview

- Definition of Subtraction
- Subtraction of Signed Numbers

Definition of Subtraction

We know from our experience with arithmetic that the subtraction $5 - 2$ produces 3, that is, $5 - 2 = 3$. Illustrating this process on the number line suggests a rule for subtracting signed numbers.



We begin at 0, the origin.
Since 5 is positive, we move 5 units to the right.
Then, we move **2 units to the left** to get to 3. (This reminds us of addition with a negative number.)

This illustration suggests that $5 - 2$ is the same as $5 + (-2)$.
This leads us directly to the definition of subtraction.

Definition of Subtraction

If a and b are real numbers, $a - b$ is the same as $a + (-b)$, where $-b$ is the opposite of b.

Subtraction of Signed Numbers

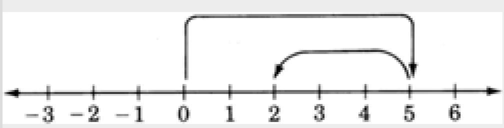
The preceding definition suggests the rule for subtracting signed numbers.

Subtraction of Signed Numbers
To perform the subtraction $a - b$, add the opposite of b to a, that is, change the sign of b and add.

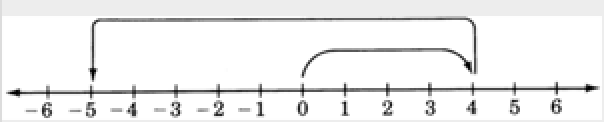
Sample Set A

Perform the subtractions.

Example:
$5 - 3 = 5 + (-3) = 2$

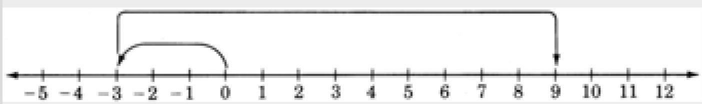


Example:
$4 - 9 = 4 + (-9) = -5$

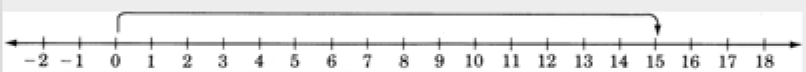


Example:
$-4-6=-4+(-6)=-10$

Example:
$-3-(-12)=-3+12=9$



Example:
$0-(-15)=0+15=15$



Example:
The high temperature today in Lake Tahoe was 26°F. The low temperature tonight is expected to be -7°F. How many degrees is the temperature expected to drop?
We need to find the difference between 26 and -7.
$26-(-7)=26+7=33$
Thus, the expected temperature drop is 33°F.

Example:

$$\begin{aligned} -6-(-5)-10 &= -6+5+(-10) \\ &= (-6+5)+(-10) \\ &= -1+(-10) \\ &= -11 \end{aligned}$$

Practice Set A

Perform the subtractions.

Exercise:

Problem: $9-6$

Solution:

3

Exercise:

Problem: $6-9$

Solution:

-3

Exercise:

Problem: $0-7$

Solution:

-7

Exercise:

Problem: $1 - 14$

Solution:

-13

Exercise:

Problem: $-8 - 12$

Solution:

-20

Exercise:

Problem: $-21 - 6$

Solution:

-27

Exercise:

Problem: $-6 - (-4)$

Solution:

-2

Exercise:

Problem: $8 - (-10)$

Solution:

18

Exercise:

Problem: $1 - (-12)$

Solution:

13

Exercise:

Problem: $86 - (-32)$

Solution:

118

Exercise:

Problem: $0 - 16$

Solution:

-16

Exercise:

Problem: $0 - (-16)$

Solution:

16

Exercise:

Problem: $0 - (8)$

Solution:

-8

Exercise:

Problem: $5-(-5)$

Solution:

10

Exercise:

Problem: $24-(-(-24))$

Solution:

0

Exercises

For the following exercises, perform the indicated operations.

Exercise:

Problem: $8-3$

Solution:

5

Exercise:

Problem: $12-7$

Exercise:

Problem: $5-6$

Solution:

-1

Exercise:

Problem: $14 - 30$

Exercise:

Problem: $2 - 15$

Solution:

-13

Exercise:

Problem: $5 - 18$

Exercise:

Problem: $1 - 7$

Solution:

-6

Exercise:

Problem: $4 - 11$

Exercise:

Problem: $-6 - 5$

Solution:

-11

Exercise:

Problem: $-8-14$

Exercise:

Problem: $-1-12$

Solution:

-13

Exercise:

Problem: $-4-4$

Exercise:

Problem: $-6-8$

Solution:

-14

Exercise:

Problem: $-1-12$

Exercise:

Problem: $-5-(-3)$

Solution:

-2

Exercise:

Problem: $-11-(-8)$

Exercise:

Problem: $-7-(-12)$

Solution:

5

Exercise:

Problem: $-2-(-10)$

Exercise:

Problem: $-4-(-15)$

Solution:

11

Exercise:

Problem: $-11-(-16)$

Exercise:

Problem: $-1-(-6)$

Solution:

5

Exercise:

Problem: $-8-(-14)$

Exercise:

Problem: $-15-(-10)$

Solution:

-5

Exercise:

Problem: $-11-(-4)$

Exercise:

Problem: $-16-(-8)$

Solution:

-8

Exercise:

Problem: $-12-(-11)$

Exercise:

Problem: $0-6$

Solution:

-6

Exercise:

Problem: $0-15$

Exercise:

Problem: $0 - (-7)$

Solution:

7

Exercise:

Problem: $0 - (-10)$

Exercise:

Problem: $67 - 38$

Solution:

29

Exercise:

Problem: $142 - 85$

Exercise:

Problem: $816 - 1140$

Solution:

-324

Exercise:

Problem: $105 - 421$

Exercise:

Problem: $-550 - (-121)$

Solution:

-429

Exercise:

Problem: $-15.016-(4.001)$

Exercise:

Problem: $-26+7-52$

Solution:

-71

Exercise:

Problem: $-15-21-(-2)$

Exercise:

Problem: $-104-(-216)-(-52)$

Solution:

164

Exercise:

Problem: $-0.012-(-0.111)-(0.035)$

Exercise:

Problem: $[5+(-6)]-[2+(-4)]$

Solution:

1

Exercise:

Problem: $[2+(-8)]-[5+(-7)]$

Exercise:

Problem: $[4+(-11)]-[2+(-10)]$

Solution:

1

Exercise:

Problem: $[9+(-6)]-[4+(-12)]$

Exercise:

Problem: $(11-8)-(1-6)$

Solution:

8

Exercise:

Problem: $(5-12)-(4-10)$

Exercise:

Problem: $(1-10)-(2-15)$

Solution:

4

Exercise:

Problem: $(0-8)-(4-12)$

Exercise:

Problem: $(-4+7)-(2-5)$

Solution:

6

Exercise:

Problem: $(-6+2)-(5-11)$

Exercise:

Problem: $[-8+(-5+3)]-[9-(-3-5)]$

Solution:

-27

Exercise:

Problem: $[-4+(-1+6)]-[7-(-6-1)]$

Exercise:

Problem: $[2-(-6+10)]-[1-(2-11)]$

Solution:

-12

Exercise:

Problem: $[5-(-2-5)]-[2-(-1-4)]$

Exercise:

Problem:

When a particular machine is operating properly, its meter will read 34. If a broken bearing in the machine causes the meter reading to drop by 45 units, what is the meter reading?

Solution:

-11

Exercise:

Problem:

The low temperature today in Denver was $-4°\text{F}$ and the high was $42°\text{F}$. What is the temperature difference?

Exercises for Review

Exercise:

Problem: ([link]) Find the sum. $-8+(-14)$.

Exercise:

Problem: ([link]) Find the sum. $3+(-6)$.

Solution:

-3

Basic Operations with Real Numbers: Multiplication and Division of Signed Numbers This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The basic operations with real numbers are presented in this chapter. The concept of absolute value is discussed both geometrically and symbolically. The geometric presentation offers a visual understanding of the meaning of |x|. The symbolic presentation includes a literal explanation of how to use the definition. Negative exponents are developed, using reciprocals and the rules of exponents the student has already learned. Scientific notation is also included, using unique and real-life examples. Objectives of this module: be able to multiply and divide signed numbers.

Overview

- Multiplication of Signed Numbers
- Division of Signed Numbers

Multiplication of Signed Numbers

Let us consider first the product of two positive numbers.

Multiply: $3 \cdot 5$.
$3 \cdot 5$ means $5+5+5=15$.

This suggests that

$(\text{positive number}) \cdot (\text{positive number}) = \text{positive number}$.

More briefly, $(+)(+)=+$.

Now consider the product of a positive number and a negative number.

Multiply: $(3)(-5)$.
$(3)(-5)$ means $(-5)+(-5)+(-5)=-15$.

This suggests that

$(\text{positive number}) \cdot (\text{negative number}) = \text{negative number}$

More briefly, $(+)(-)=-$.

By the commutative property of multiplication, we get

$(\text{negative number}) \cdot (\text{positive number}) = \text{negative number}$

More briefly, $(-)(+) = -$.

The sign of the product of two negative numbers can be determined using the following illustration: Multiply -2 by, respectively, $4,\ 3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3,\ -4$. Notice that when the multiplier decreases by 1, the product increases by 2.

$$\left.\begin{array}{l} 4(-2) = -8 \\ 3(-2) = -6 \\ 2(-2) = -4 \\ 1(-2) = -2 \end{array}\right\} \rightarrow \quad \text{As we know, } (+)(-) = -.$$

$$0(-2) = 0 \quad \rightarrow \quad \text{As we know, } 0 \cdot (\text{any number}) = 0.$$

$$\left.\begin{array}{l} -1(-2) = 2 \\ -2(-2) = 4 \\ -3(-2) = 6 \\ -4(-2) = 8 \end{array}\right\} \rightarrow \quad \text{This pattern suggests } (-)(-) = +.$$

We have the following rules for multiplying signed numbers.

Rules for Multiplying Signed Numbers
To multiply two real numbers that have

1. the **same sign**, multiply their absolute values. The product is positive.
   $(+)(+) = +$
   $(-)(-) = +$
2. **opposite signs**, multiply their absolute values. The product is negative.
   $(+)(-) = -$
   $(-)(+) = -$

Sample Set A

Find the following products.

Example:
$8 \cdot 6$

Multiply these absolute values.

$$\left.\begin{aligned}|8| = 8\\ |6| = 6\end{aligned}\right\} \quad 8 \cdot 6 = 48$$

Since the numbers have the same sign, the product is positive.

$8 \cdot 6 = +48$ or $8 \cdot 6 = 48$

Example:

$(-8)(-6)$

Multiply these absolute values.

$$\left.\begin{aligned}|-8| = 8\\ |-6| = 6\end{aligned}\right\} \quad 8 \cdot 6 = 48$$

Since the numbers have the same sign, the product is positive.

$(-8)(-6) = +48$ or $(-8)(-6) = 48$

Example:

$(-4)(7)$

Multiply these absolute values.

$$\left.\begin{aligned}|-4| &= 4\\ |7| &= 7\end{aligned}\right\} \quad 4 \cdot 7 = 28$$

Since the numbers have opposite signs, the product is negative.

$(-4)(7) = -28$

Example:

$6(-3)$

Multiply these absolute values.

$$\left.\begin{aligned}|6| &= 6\\ |-3| &= 3\end{aligned}\right\} \quad 6 \cdot 3 = 18$$

Since the numbers have opposite signs, the product is negative.

$6(-3) = -18$

Practice Set A

Find the following products.

Exercise:

Problem: $3(-8)$

Solution:

-24

Exercise:

Problem: $4(16)$

Solution:

64

Exercise:

Problem: $(-6)(-5)$

Solution:

30

Exercise:

Problem: $(-7)(-2)$

Solution:

14

Exercise:

Problem: $(-1)(4)$

Solution:

-4

Exercise:

Problem: $(-7)7$

Solution:

-49

Division of Signed Numbers

We can determine the sign pattern for division by relating division to multiplication. Division is defined in terms of multiplication in the following way.

If $b \cdot c = a$, then $\frac{a}{b} = c$, $b \neq 0$.

For example, since $3 \cdot 4 = 12$, it follows that $\frac{12}{3} = 4$.

Notice the pattern:

Since $\underbrace{3 \cdot 4 = 12}_{b \cdot c = a}$, it follows that $\underbrace{\frac{12}{3} = 4}_{\frac{a}{b} = c}$

The sign pattern for division follows from the sign pattern for multiplication.

1. Since $\underbrace{(+)(+) = +}_{b \cdot c = a}$, it follows that $\underbrace{\frac{(+)}{(+)} = +}_{\frac{a}{b} = c}$, that is,

$$\frac{(\text{positive number})}{(\text{positive number})} = \text{positive number}$$

2. Since $\underbrace{(-)(-) = +}_{b \cdot c = a}$, it follows that $\underbrace{\frac{(+)}{(-)} = -}_{\frac{a}{b} = c}$, that is,

$$\frac{(\text{positive number})}{(\text{negative number})} = \text{negative number}$$

3. Since $\underbrace{(+)(-) = -}_{b \cdot c = a}$, it follows that $\underbrace{\frac{(-)}{(+)} = -}_{\frac{a}{b} = c}$, that is,

$$\frac{(\text{negative number})}{(\text{positive number})} = \text{negative number}$$

4. Since $\underbrace{(-)(+) = -}_{b \cdot c = a}$, it follows that $\underbrace{\frac{(-)}{(-)} = +}_{\frac{a}{b} = c}$, that is

$$\frac{(\text{negative number})}{(\text{negative number})} = \text{positive number}$$

We have the following rules for dividing signed numbers.

Rules for Dividing Signed Numbers
To divide two real numbers that have

1. the **same sign**, divide their absolute values. The quotient is positive.
   $\frac{(+)}{(+)} = + \quad \frac{(-)}{(-)} = +$
2. **opposite signs**, divide their absolute values. The quotient is negative.
   $\frac{(-)}{(+)} = - \quad \frac{(+)}{(-)} = -$

Sample Set B

Find the following quotients.

Example:
$\frac{-10}{2}$

$$\begin{aligned} \left.\begin{aligned} |-10| &= 10 \\ |2| &= 2 \end{aligned}\right\} & \quad \text{Divide these absolute values.} \\ & \quad \frac{10}{2} = 5 \\ \frac{-10}{2} = -5 & \quad \text{Since the numbers have opposite signs, the quotient is negative.} \end{aligned}$$

Example:
$\frac{-35}{-7}$

$$\begin{aligned} \left.\begin{aligned} |-35| &= 35 \\ |-7| &= 7 \end{aligned}\right\} & \quad \text{Divide these absolute values.} \\ & \quad \frac{35}{7} = 5 \\ \frac{-35}{-7} = 5 & \quad \text{Since the numbers have same signs, the quotient is positive.} \end{aligned}$$

Example:
$\frac{18}{-9}$

$$\left.\begin{array}{rl}|18| = & 18 \\ |-9| = & 9\end{array}\right\} \text{ Divide these absolute values.}$$

$$\frac{18}{9} = 2$$

$$\frac{18}{-9} = -2 \quad \text{Since the numbers have opposite signs, the quotient is negative.}$$

Practice Set B

Find the following quotients.

Exercise:

Problem: $\frac{-24}{-6}$

Solution:

4

Exercise:

Problem: $\frac{30}{-5}$

Solution:

-6

Exercise:

Problem: $\frac{-54}{27}$

Solution:

-2

Exercise:

Problem: $\frac{51}{17}$

Solution:

3

Sample Set C

Example:
Find the value of $\frac{-6(4-7)-2(8-9)}{-(4+1)+1}$.
Using the order of operations and what we know about signed numbers, we get

$$\begin{aligned} \frac{-6(4-7)-2(8-9)}{-(4+1)+1} &= \frac{-6(-3)-2(-1)}{-(5)+1} \\ &= \frac{18+2}{-5+1} \\ &= \frac{20}{-4} \\ &= -5 \end{aligned}$$

Example:
Find the value of $z = \frac{x-u}{s}$ if $x = 57,\ u = 51,$ and $s = 2$.
Substituting these values we get
$z = \frac{57-51}{2} = \frac{6}{2} = 3$

Practice Set C

Exercise:

Problem: Find the value of $\frac{-7(4-8)+2(1-11)}{-5(1-6)-17}$.

Solution:

1

Exercise:

Problem: Find the value of $P = \frac{n(n-3)}{2n}$, if $n = 5$.

Solution:

1

Exercises

Find the value of each of the following expressions.

Exercise:

Problem: $(-2)(-8)$

Solution:

16

Exercise:

Problem: $(-3)(-9)$

Exercise:

Problem: $(-4)(-8)$

Solution:

32

Exercise:

Problem: $(-5)(-2)$

Exercise:

Problem: $(-6)(-9)$

Solution:

54

Exercise:

Problem: $(-3)(-11)$

Exercise:

Problem: $(-8)(-4)$

Solution:

32

Exercise:

Problem: $(-1)(-6)$

Exercise:

Problem: $(3)(-12)$

Solution:

-36

Exercise:

Problem: $(4)(-18)$

Exercise:

Problem: $8(-4)$

Solution:

-32

Exercise:

Problem: $5(-6)$

Exercise:

Problem: $9(-2)$

Solution:

-18

Exercise:

Problem: $7(-8)$

Exercise:

Problem: $(-6)4$

Solution:

-24

Exercise:

Problem: $(-7)6$

Exercise:

Problem: $(-10)9$

Solution:

-90

Exercise:

Problem: $(-4)12$

Exercise:

Problem: $(10)(-6)$

Solution:

-60

Exercise:

Problem: $(-6)(4)$

Exercise:

Problem: $(-2)(6)$

Solution:

-12

Exercise:

Problem: $(-8)(7)$

Exercise:

Problem: $\frac{21}{7}$

Solution:

3

Exercise:

Problem: $\frac{42}{6}$

Exercise:

Problem: $\frac{-39}{3}$

Solution:

-13

Exercise:

Problem: $\frac{-20}{10}$

Exercise:

Problem: $\frac{-45}{-5}$

Solution:

9

Exercise:

Problem: $\frac{-16}{-8}$

Exercise:

Problem: $\frac{25}{-5}$

Solution:

-5

Exercise:

Problem: $\frac{36}{-4}$

Exercise:

Problem: $8-(-3)$

Solution:

11

Exercise:

Problem: $14-(-20)$

Exercise:

Problem: $20-(-8)$

Solution:

28

Exercise:

Problem: $-4-(-1)$

Exercise:

Problem: $0-4$

Solution:

-4

Exercise:

Problem: $0-(-1)$

Exercise:

Problem: $-6+1-7$

Solution:

-12

Exercise:

Problem: $15-12-20$

Exercise:

Problem: $1-6-7+8$

Solution:

-4

Exercise:

Problem: $2+7-10+2$

Exercise:

Problem: $3(4-6)$

Solution:

-6

Exercise:

Problem: $8(5-12)$

Exercise:

Problem: $-3(1-6)$

Solution:

15

Exercise:

Problem: $-8(4-12)+2$

Exercise:

Problem: $-4(1-8)+3(10-3)$

Solution:

49

Exercise:

Problem: $-9(0-2)+4(8-9)+0(-3)$

Exercise:

Problem: $6(-2-9)-6(2+9)+4(-1-1)$

Solution:

-140

Exercise:

Problem: $\frac{3(4+1)-2(5)}{-2}$

Exercise:

Problem: $\frac{4(8+1)-3(-2)}{-4-2}$

Solution:

-7

Exercise:

Problem: $\frac{-1(3+2)+5}{-1}$

Exercise:

Problem: $\frac{-3(4-2)+(-3)(-6)}{-4}$

Solution:

-3

Exercise:

Problem: $-1(4+2)$

Exercise:

Problem: $-1(6-1)$

Solution:

-5

Exercise:

Problem: $-(8+21)$

Exercise:

Problem: $-(8-21)$

Solution:

13

Exercise:

Problem: $-(10-6)$

Exercise:

Problem: $-(5-2)$

Solution:

-3

Exercise:

Problem: $-(7-11)$

Exercise:

Problem: $-(8-12)$

Solution:

4

Exercise:

Problem: $-3[(-1+6)-(2-7)]$

Exercise:

Problem: $-2[(4-8)-(5-11)]$

Solution:

-4

Exercise:

Problem: $-5[(-1+5)+(6-8)]$

Exercise:

Problem: $-[(4-9)+(-2-8)]$

Solution:

15

Exercise:

Problem: $-3[-2(1-5)-3(-2+6)]$

Exercise:

Problem: $-2[-5(-10+11)-2(5-7)]$

Solution:

2

Exercise:

Problem: $P = R - C$. Find P if $R = 2000$ and $C = 2500$.

Exercise:

Problem: $z = \frac{x-u}{s}$. Find z if $x = 23,\ u = 25,$ and $s = 1$.

Solution:

-2

Exercise:

Problem: $z = \frac{x-u}{s}$. Find z if $x = 410,\ u = 430,$ and $s = 2.5$.

Exercise:

Problem: $m = \frac{2s+1}{T}$. Find m if $s = -8$ and $T = 5$.

Solution:

-3

Exercise:

Problem: $m = \frac{2s+1}{T}$. Find m if $s = -10$ and $T = -5$.

Exercise:

Problem: $F = (p_1 - p_2)r^4 \cdot 9$. Find F if $p_1 = 10,\ p_2 = 8, r = 3$.

Solution:

1458

Exercise:

Problem: $F = (p_1 - p_2)r^4 \cdot 9. \quad \text{Find } F \text{ if } p_1 = 12,\ p_2 = 7, r = 2.$

Exercise:

Problem: $P = n(n-1)(n-2). \quad \text{Find } P \text{ if } n = -4.$

Solution:

-120

Exercise:

Problem: $P = n(n-1)(n-2)(n-3). \quad \text{Find } P \text{ if } n = -5.$

Exercise:

Problem: $P = \frac{n(n-2)(n-4)}{2n}. \quad \text{Find } P \text{ if } n = -6.$

Solution:

40

Exercises for Review

Exercise:

Problem:

([link]) What natural numbers can replace x so that the statement $-4 < x \le 3$ is true?

Exercise:

Problem: **([link])** Simplify $\frac{(x+2y)^5(3x-1)^7}{(x+2y)^3(3x-1)^6}$.

Solution:

$(x+2y)^2(3x-1)$

Exercise:

Problem: **([link])** Simplify $\left(x^n y^{3t}\right)^5$.

Exercise:

Problem: **([link])** Find the sum. $-6+(-5)$.

Solution:

-11

Exercise:

Problem: **([link])** Find the difference. $-2-(-8)$.

Basic Operations with Real Numbers: Summary of Key Concepts
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The basic operations with real numbers are presented in this chapter. The concept of absolute value is discussed both geometrically and symbolically. The geometric presentation offers a visual understanding of the meaning of |x|. The symbolic presentation includes a literal explanation of how to use the definition. Negative exponents are developed, using reciprocals and the rules of exponents the student has already learned. Scientific notation is also included, using unique and real-life examples. This module contains a summary of the key concepts in the chapter "Basic Operations with Real Numbers".

Summary of Key Concepts

Positive and Negative Numbers ([link])
A number is denoted as **positive** if it is directly preceded by a "$+$" sign or no sign at all. A number is denoted as negative if it is directly preceded by a "$-$" sign.

Opposites ([link])
Opposites are numbers that are the same distance from zero on the number line but have opposite signs.

Double-Negative Property ([link])
$-(-a) = a$

Absolute Value (Geometric) ([link])
The absolute value of a number a, denoted $|a|$, is the distance from a to 0 on the number line.

Absolute Value (Algebraic) ([link])
$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Addition of Signed Numbers ([link])
To add two numbers with
like signs, add the absolute values of the numbers and associate the

common sign with the sum.
unlike signs, subtract the smaller absolute value from the larger absolute value and associate the sign of the larger absolute value with the difference.

Addition with 0 ([link])
$0 + \text{any number} = \text{that particular number}$, that is, $0 + a = a$ for any real number a.

Additive Identity ([link])
Since adding 0 to a real number leaves that number unchanged, 0 is called the additive identity.

Definition of Subtraction ([link])
$a - b = a + (-b)$

Subtraction of Signed Numbers ([link])
To perform the subtraction $a - b$, add the opposite of b to a, that is, change the sign of b and add.

Multiplication and Division of Signed Numbers ([link])
$(+)(+) = + \quad \frac{(+)}{(+)} = + \quad \frac{(+)}{(-)} = -$
$(-)(-) = +$
$(+)(-) = -$
$(-)(+) = - \quad \frac{(-)}{(-)} = + \quad \frac{(-)}{(+)} = -$

Reciprocals ([link])
Two numbers are reciprocals of each other if their product is 1. The numbers 4 and $\frac{1}{4}$ are reciprocals since $(4)\left(\frac{1}{4}\right) = 1$.

Negative Exponents ([link])
If n is any natural number and x is any nonzero real number, then $x^{-n} = \frac{1}{x^n}$.

Writing a Number in Scientific Notation ([link])
To write a number in scientific notation:

1. Move the decimal point so that there is one nonzero digit to its left.
2. Multiply the result by a power of 10 using an exponent whose absolute value is the number of places the decimal point was moved. Make the exponent positive if the decimal point was moved to the left and negative if the decimal point was moved to the right.

Converting from Scientific Notation:
positive exponent ([link])
To convert a number written in scientific notation to a number in standard form when there is a **positive** exponent as the power of 10, move the decimal point to the **right** the number of places prescribed by the exponent on the 10.

Negative Exponent ([link])
To convert a number written in scientific notation to a number in standard form when there is a **negative** exponent as the power of 10, move the decimal point to the **left** the number of places prescribed by the exponent on the 10.

Use the Rectangular Coordinate System
By the end of this section, you will be able to:

- Plot points on a rectangular coordinate system
- Identify points on a graph
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to linear equations in two variables

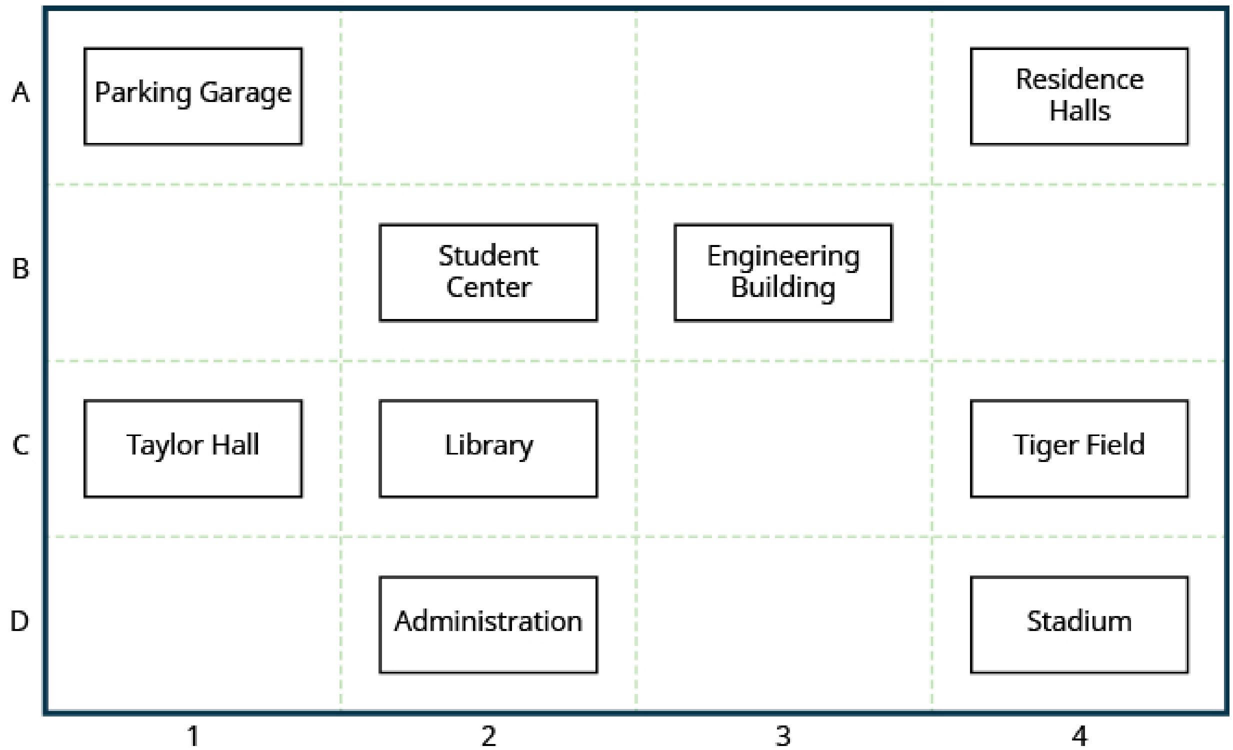
Note:
Before you get started, take this readiness quiz.

1. Evaluate: $x+3$ when $x=-1$.
   If you missed this problem, review [link].
2. Evaluate: $2x-5y$ when $x=3, y=-2$.
   If you missed this problem, review [link].
3. Solve for y: $40-4y=20$.
   If you missed this problem, review [link].

Plot Points on a Rectangular Coordinate System

Many maps, such as the Campus Map shown in [link], use a grid system to identify locations. Do you see the numbers $1, 2, 3$, and 4 across the top and bottom of the map and the letters A, B, C, and D along the sides? Every location on the map can be identified by a number and a letter.

For example, the Student Center is in section 2B. It is located in the grid section above the number 2 and next to the letter B. In which grid section is the Stadium? The Stadium is in section 4D.



Example:
Exercise:

Problem: Use the map in [link].

ⓐ Find the grid section of the Residence Halls.
ⓑ What is located in grid section 4C?

Solution:
Solution

ⓐ Read the number below the Residence Halls, 4, and the letter to the side, A. So the Residence Halls are in grid section 4A.
ⓑ Find 4 across the bottom of the map and C along the side. Look below the 4 and next to the C. Tiger Field is in grid section 4C.

Note:
Exercise:

Problem: Use the map in [link].

ⓐ Find the grid section of Taylor Hall.
ⓑ What is located in section 3B?

Solution:

ⓐ 1C
ⓑ Engineering Building

Note:
Exercise:

Problem: Use the map in [link].

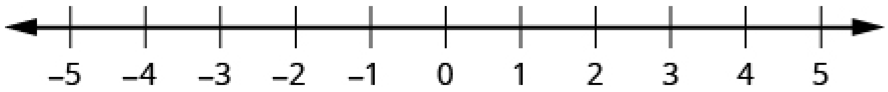
ⓐ Find the grid section of the Parking Garage.
ⓑ What is located in section 2C?

Solution:

ⓐ 1A
ⓑ Library

Just as maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. To create a rectangular coordinate system, start with a

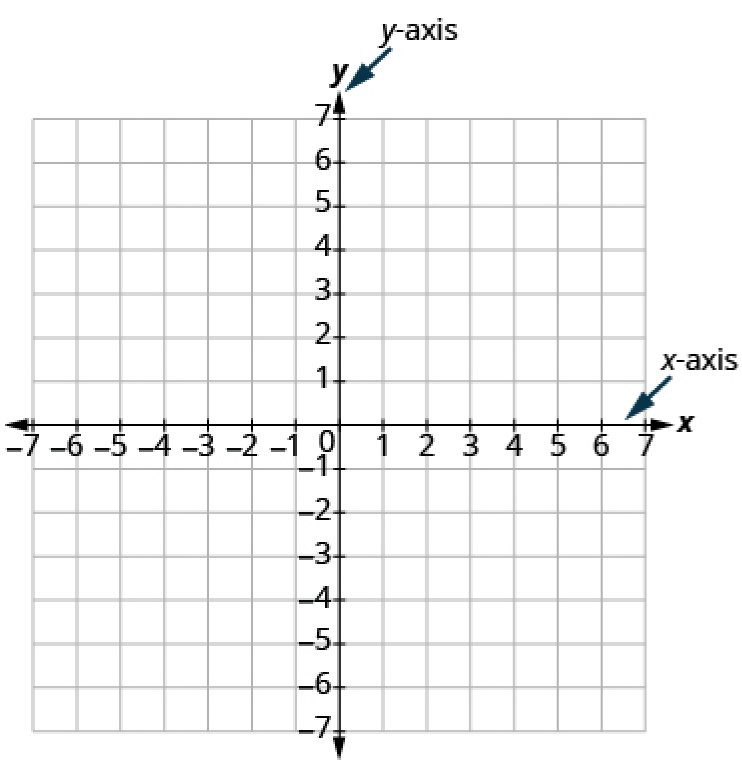
horizontal number line. Show both positive and negative numbers as you did before, using a convenient scale unit. This horizontal number line is called the ***x*-axis**.



Now, make a vertical number line passing through the x-axis at 0. Put the positive numbers above 0 and the negative numbers below 0. See [link]. This vertical line is called the ***y*-axis**.

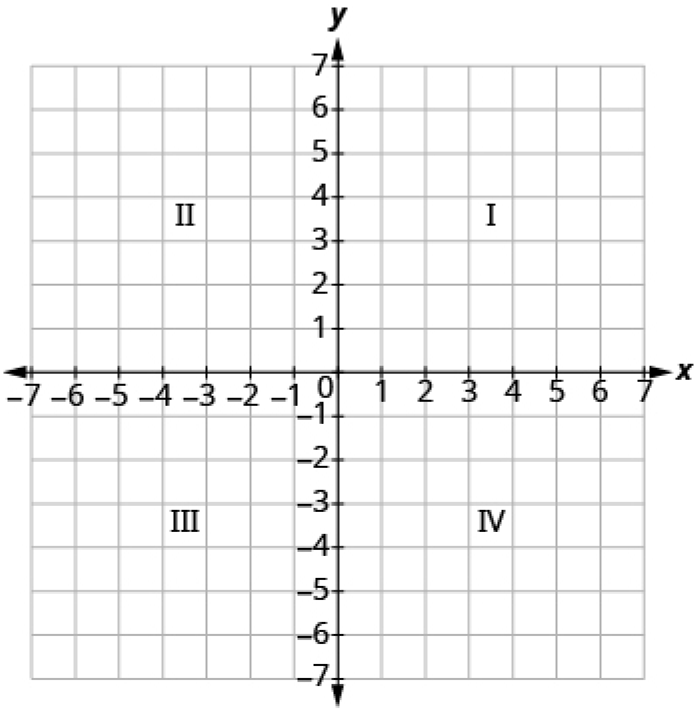
Vertical grid lines pass through the integers marked on the x-axis. Horizontal grid lines pass through the integers marked on the y-axis. The resulting grid is the rectangular coordinate system.

The rectangular coordinate system is also called the x-y plane, the coordinate plane, or the Cartesian coordinate system (since it was developed by a mathematician named René Descartes.)



The rectangular coordinate system.

The x-axis and the y-axis form the rectangular coordinate system. These axes divide a plane into four areas, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See [link].

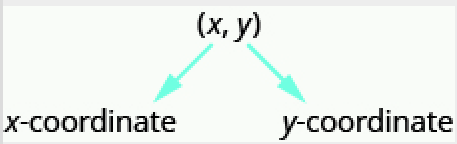


The four quadrants of the rectangular coordinate system

In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the x-coordinate of the point, and the second number is the y-coordinate of the point.

Note:
Ordered Pair
An ordered pair, (x,y) gives the coordinates of a point in a rectangular coordinate system.
Equation:

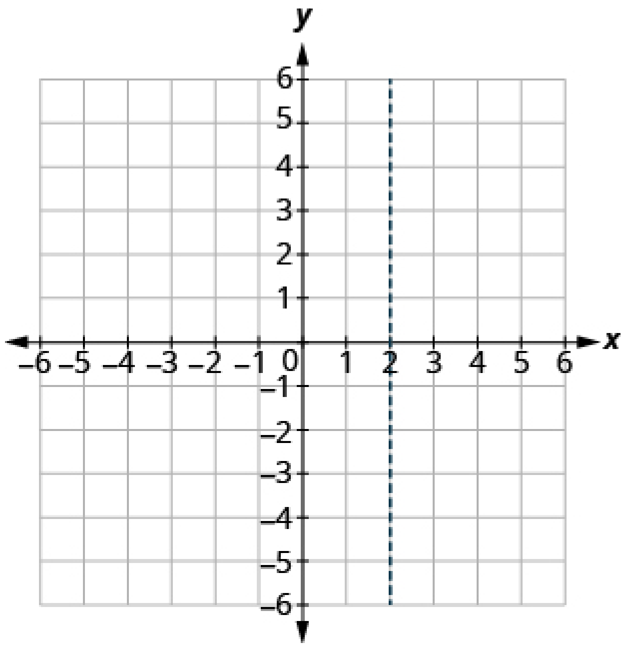
$$\begin{array}{l}\text{The first number is the } x\text{-coordinate.}\\ \text{The second number is the } y\text{-coordinate.}\end{array}$$



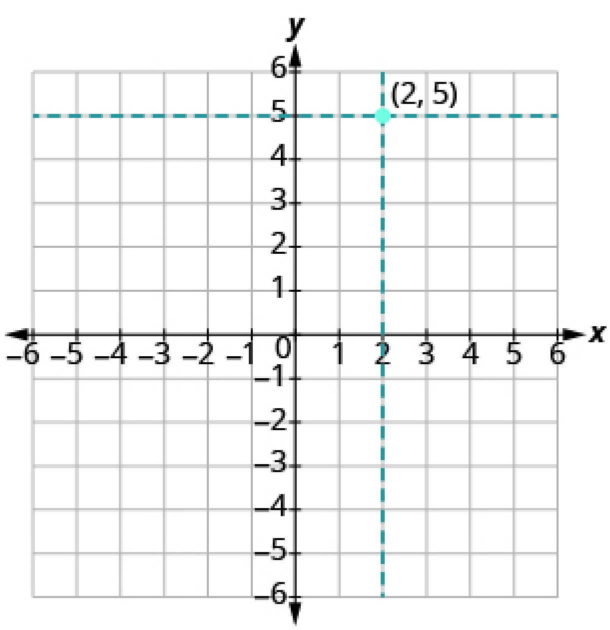
So how do the coordinates of a point help you locate a point on the x-y plane?

Let’s try locating the point $(2,5)$. In this ordered pair, the x-coordinate is 2 and the y-coordinate is 5.

We start by locating the x value, 2, on the x-axis. Then we lightly sketch a vertical line through $x=2$, as shown in [link].



Now we locate the y value, 5, on the y-axis and sketch a horizontal line through $y = 5$. The point where these two lines meet is the point with coordinates $(2, 5)$. We plot the point there, as shown in [link].



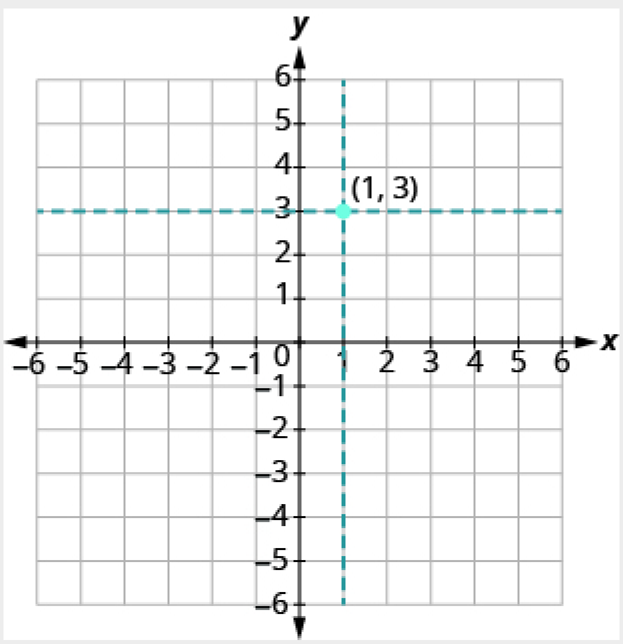
Example:
Exercise:

Problem:

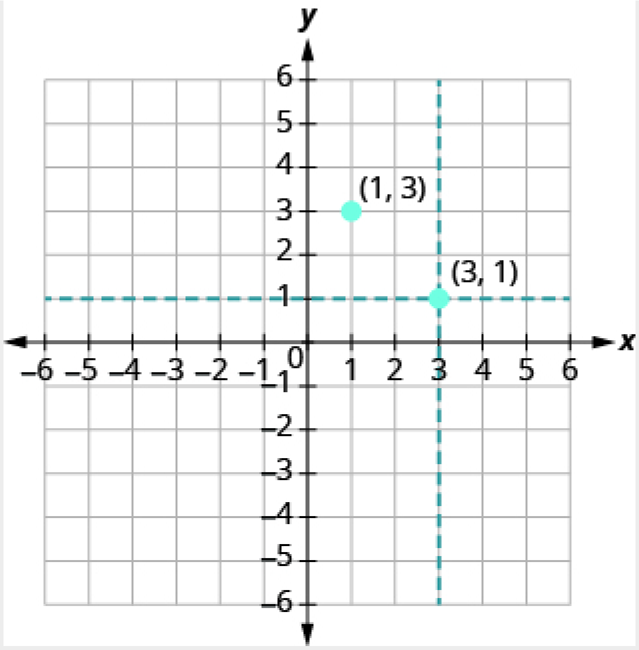
Plot $(1, 3)$ and $(3, 1)$ in the same rectangular coordinate system.

Solution:
Solution

The coordinate values are the same for both points, but the x and y values are reversed. Let's begin with point $(1, 3)$. The x-coordinate is 1 so find 1 on the x-axis and sketch a vertical line through $x = 1$. The y-coordinate is 3 so we find 3 on the y-axis and sketch a horizontal line through $y = 3$. Where the two lines meet, we plot the point $(1, 3)$.



To plot the point $(3, 1)$, we start by locating 3 on the x-axis and sketch a vertical line through $x = 3$. Then we find 1 on the y-axis and sketch a horizontal line through $y = 1$. Where the two lines meet, we plot the point $(3, 1)$.



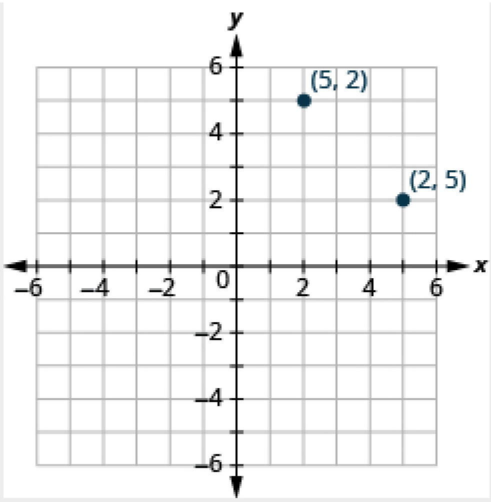
Notice that the order of the coordinates does matter, so, $(1,3)$ is not the same point as $(3,1)$.

Note:
Exercise:

Problem:

Plot each point on the same rectangular coordinate system: $(2,5),(5,2)$.

Solution:

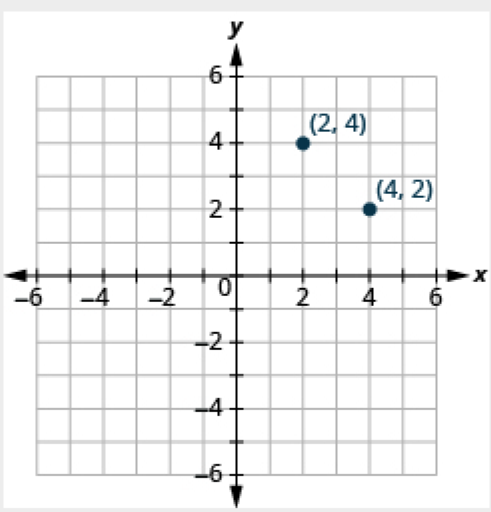


Note:
Exercise:

Problem:

Plot each point on the same rectangular coordinate system:
$(4,2),(2,4)$.

Solution:



Example:
Exercise:

Problem:

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-1, 3)$
ⓑ $(-3, -4)$
ⓒ $(2, -3)$
ⓓ $\left(3, \frac{5}{2}\right)$

Solution:
Solution

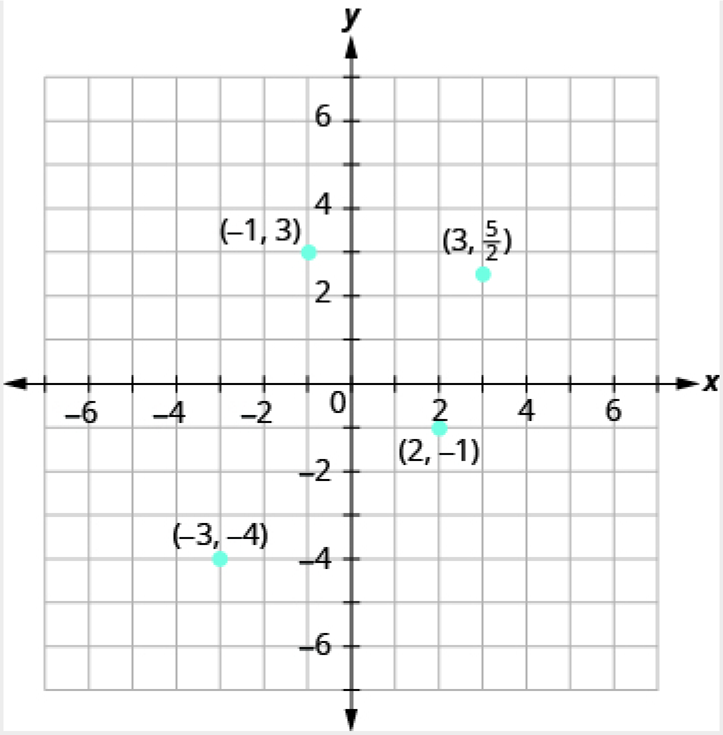
The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate.

ⓐ Since $x = -1, y = 3$, the point $(-1, 3)$ is in Quadrant II.

ⓑ Since $x = -3, y = -4$, the point $(-3, -4)$ is in Quadrant III.

ⓒ Since $x = 2, y = -1$, the point $(2, -1)$ is in Quadrant lV.

ⓓ Since $x = 3, y = \frac{5}{2}$, the point $\left(3, \frac{5}{2}\right)$ is in Quadrant I. It may be helpful to write $\frac{5}{2}$ as the mixed number, $2\frac{1}{2}$, or decimal, 2.5. Then we know that the point is halfway between 2 and 3 on the y-axis.



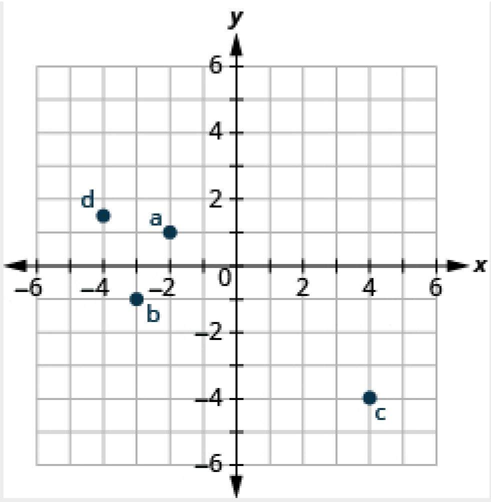
Note:
Exercise:

Problem:

Plot each point on a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-2, 1)$
ⓑ $(-3, -1)$
ⓒ $(4, -4)$
ⓓ $\left(-4, \frac{3}{2}\right)$

Solution:



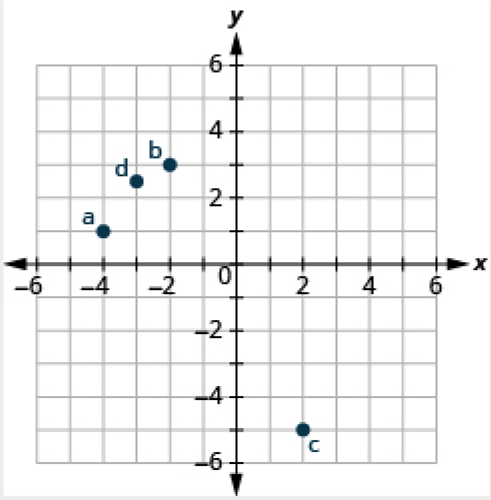
Note:
Exercise:

Problem:

Plot each point on a rectangular coordinate system and identify the quadrant in which the point is located

ⓐ $(-4, 1)$
ⓑ $(-2, 3)$
ⓒ $(2, -5)$
ⓓ $\left(-3, \frac{5}{2}\right)$

Solution:



How do the signs affect the location of the points?

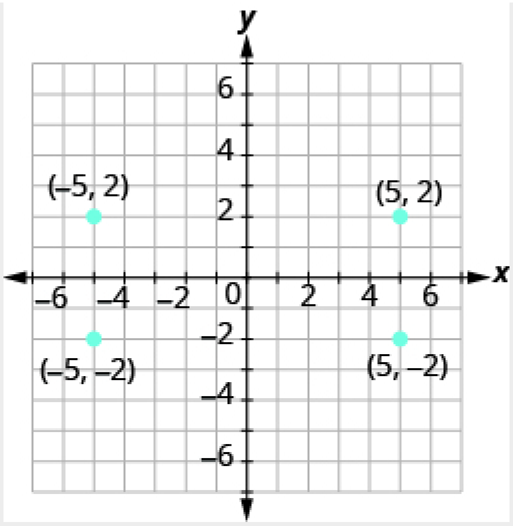
Example:
Exercise:

Problem: Plot each point:

ⓐ $(-5, 2)$
ⓑ $(-5, -2)$
ⓒ $(5, 2)$
ⓓ $(5, -2)$

Solution:
Solution

As we locate the x-coordinate and the y-coordinate, we must be careful with the signs.

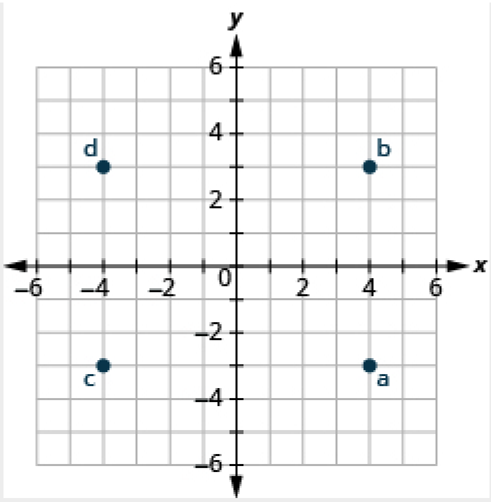


Note:
Exercise:

Problem: Plot each point:

ⓐ $(4, -3)$
ⓑ $(4, 3)$
ⓒ $(-4, -3)$
ⓓ $(-4, 3)$

Solution:

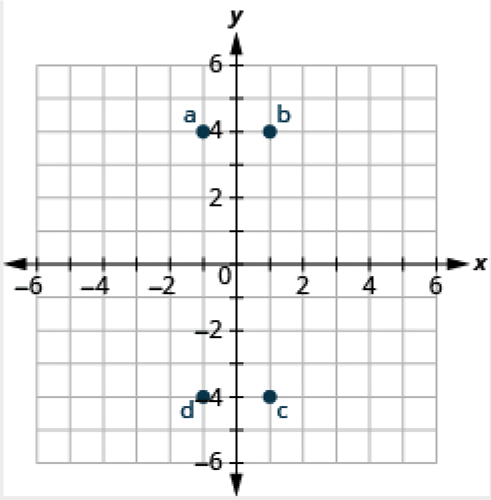


Note:
Exercise:

Problem: Plot each point:

ⓐ $(-1, 4)$
ⓑ $(1, 4)$
ⓒ $(1, -4)$
ⓓ $(-1, -4)$

Solution:

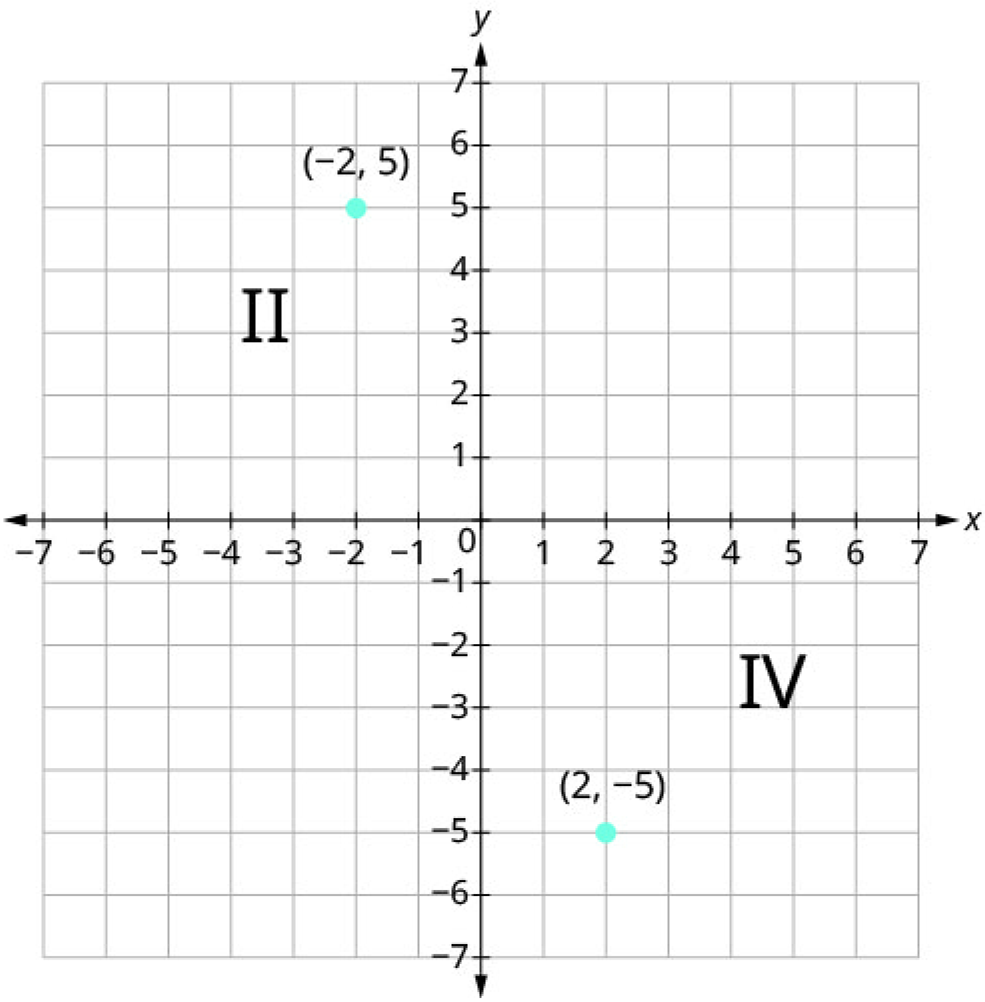


You may have noticed some patterns as you graphed the points in the two previous examples.

For each point in Quadrant IV, what do you notice about the signs of the coordinates?

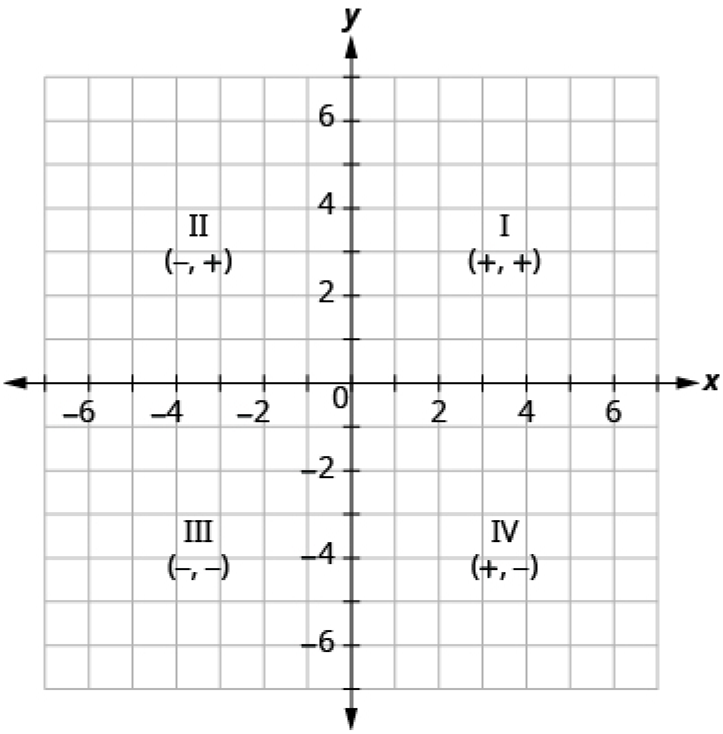
What about the signs of the coordinates of the points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point $(-2, 5)$ is located? In which quadrant is $(2, -5)$ located?



We can summarize sign patterns of the quadrants as follows. Also see [link].

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$



What if one coordinate is zero? Where is the point $(0, 4)$ located? Where is the point $(-2, 0)$ located? The point $(0, 4)$ is on the *y*-axis and the point $(-2, 0)$ is on the *x*-axis.

Note:
Points on the Axes
Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point has a special name. It is called the *origin*.

Note:
The Origin

The point $(0, 0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

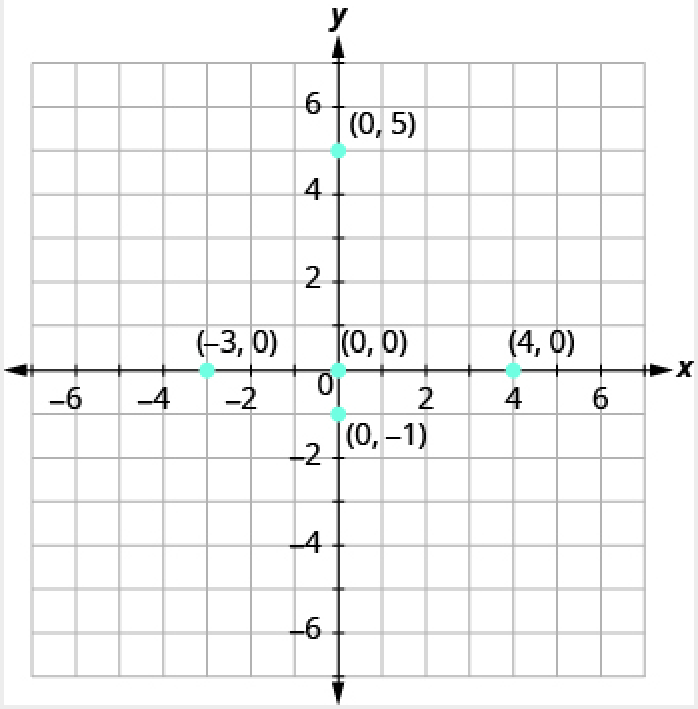
Example:
Exercise:

Problem: Plot each point on a coordinate grid:

ⓐ $(0, 5)$
ⓑ $(4, 0)$
ⓒ $(-3, 0)$
ⓓ $(0, 0)$
ⓔ $(0, -1)$

Solution:
Solution

ⓐ Since $x = 0$, the point whose coordinates are $(0, 5)$ is on the y-axis.
ⓑ Since $y = 0$, the point whose coordinates are $(4, 0)$ is on the x-axis.
ⓒ Since $y = 0$, the point whose coordinates are $(-3, 0)$ is on the x-axis.
ⓓ Since $x = 0$ and $y = 0$, the point whose coordinates are $(0, 0)$ is the origin.
ⓔ Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the y-axis.

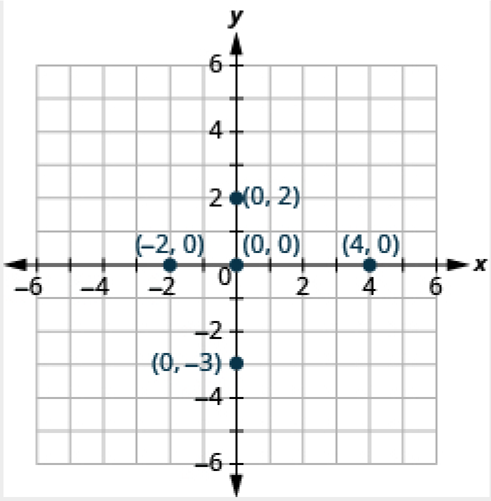


Note:
Exercise:

Problem: Plot each point on a coordinate grid:

ⓐ $(4, 0)$
ⓑ $(-2, 0)$
ⓒ $(0, 0)$
ⓓ $(0, 2)$
ⓔ $(0, -3)$

Solution:

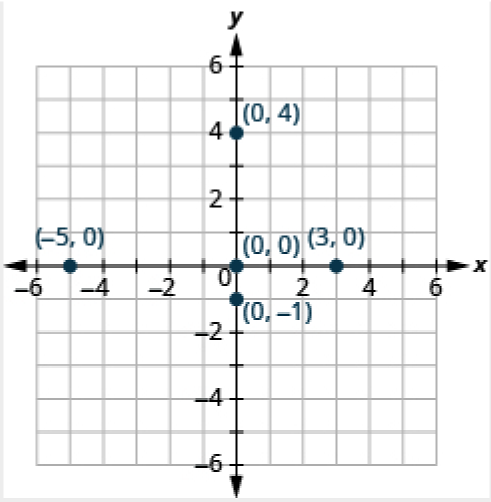


Note:
Exercise:

Problem: Plot each point on a coordinate grid:

ⓐ $(-5, 0)$
ⓑ $(3, 0)$
ⓒ $(0, 0)$
ⓓ $(0, -1)$
ⓔ $(0, 4)$

Solution:

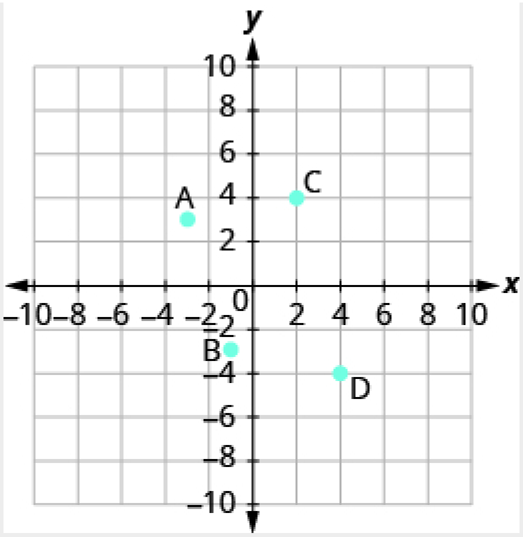


Identify Points on a Graph

In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the *x*-coordinate of a point on a graph, read the number on the *x*-axis directly above or below the point. To identify the *y*-coordinate of a point, read the number on the *y*-axis directly to the left or right of the point. Remember, to write the ordered pair using the correct order (x, y).

Example:
Exercise:

Problem: Name the ordered pair of each point shown:



Solution:
Solution

Point A is above -3 on the x-axis, so the x-coordinate of the point is -3. The point is to the left of 3 on the y-axis, so the y-coordinate of the point is 3. The coordinates of the point are $(-3, 3)$.

Point B is below -1 on the x-axis, so the x-coordinate of the point is -1. The point is to the left of -3 on the y-axis, so the y-coordinate of the point is -3. The coordinates of the point are $(-1, -3)$.

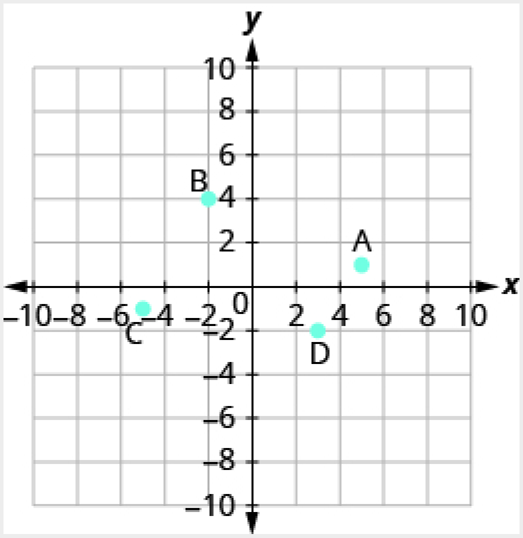
Point C is above 2 on the x-axis, so the x-coordinate of the point is 2. The point is to the right of 4 on the y-axis, so the y-coordinate of the point is 4. The coordinates of the point are $(2, 4)$.

Point D is below 4 on the $x-$ axis, so the x-coordinate of the point is 4. The point is to the right of -4 on the y-axis, so the y-coordinate of the point is -4. The coordinates of the point are $(4, -4)$.

Note:

Exercise:

Problem: Name the ordered pair of each point shown:

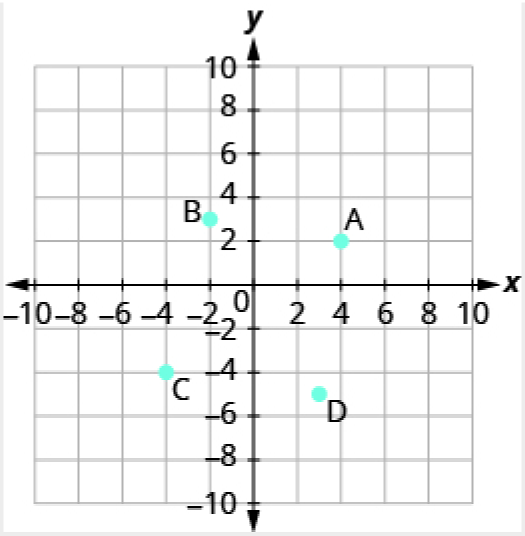


Solution:

1. A: (5,1)
2. B: (−2,4)
3. C: (−5,−1)
4. D: (3,−2)

Note:
Exercise:

Problem: Name the ordered pair of each point shown:

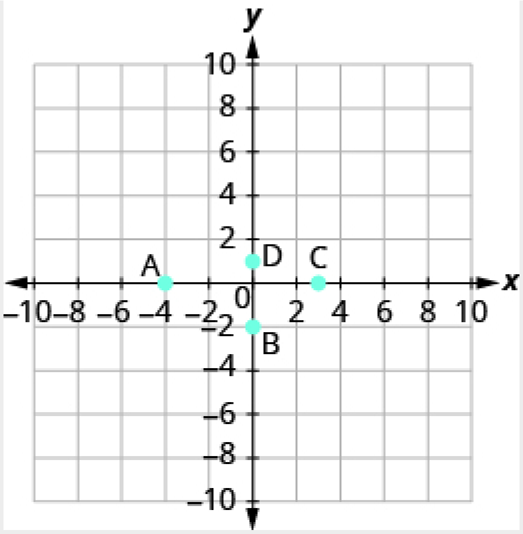


Solution:

1. A: (4,2)
2. B: (−2,3)
3. C: (−4,−4)
4. D: (3,−5)

Example:
Exercise:

Problem: Name the ordered pair of each point shown:

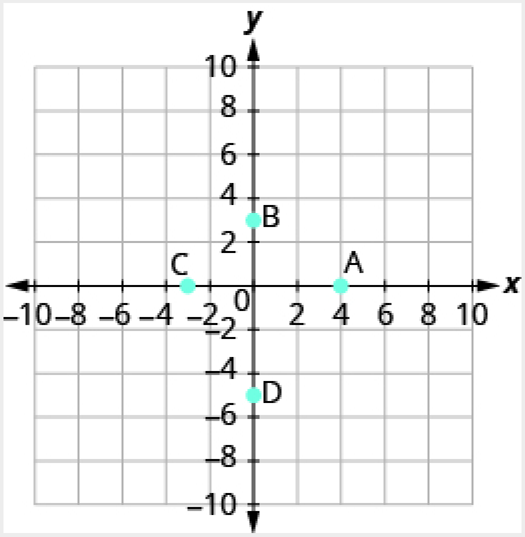


Solution:
Solution

Point A is on the *x*-axis at $x = -4$.	The coordinates of point A are $(-4, 0)$.
Point B is on the *y*-axis at $y = -2$	The coordinates of point B are $(0, -2)$.
Point C is on the *x*-axis at $x = 3$.	The coordinates of point C are $(3, 0)$.
Point D is on the *y*-axis at $y = 1$.	The coordinates of point D are $(0, 1)$.

Note:
Exercise:

Problem: Name the ordered pair of each point shown:

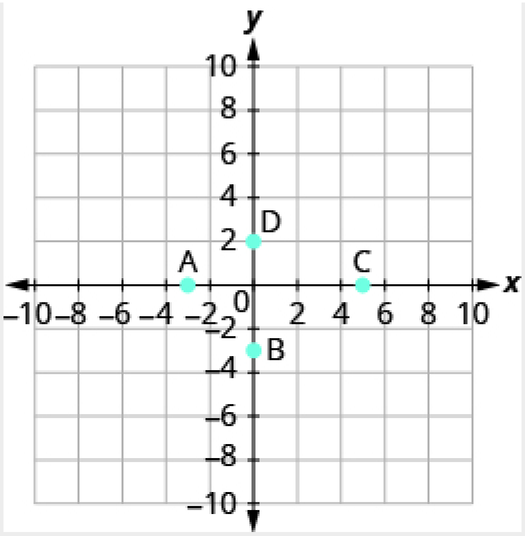


Solution:

1. A: (4,0)
2. B: (0,3)
3. C: (−3,0)
4. D: (0,−5)

Note:
Exercise:

Problem: Name the ordered pair of each point shown:



Solution:

1. A: (−3,0)
2. B: (0,−3)
3. C: (5,0)
4. D: (0,2)

Verify Solutions to an Equation in Two Variables

All the equations we solved so far have been equations with one variable. In almost every case, when we solved the equation we got exactly one solution. The process of solving an equation ended with a statement such as $x=4$. Then we checked the solution by substituting back into the equation.

Here's an example of a linear equation in one variable, and its one solution.
Equation:

$$3x+5=17$$

$$3x=12$$

$$x=4$$

But equations can have more than one variable. Equations with two variables can be written in the general form $Ax + By = C$. An equation of this form is called a linear equation in two variables.

Note:
Linear Equation
An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

Notice that the word “line” is in linear.

Here is an example of a linear equation in two variables, x and y:

$$Ax + By = C$$

$$x + 4y = 8$$

$A = 1$, $B = 4$, $C = 8$

Is $y = -5x + 1$ a linear equation? It does not appear to be in the form $Ax + By = C$. But we could rewrite it in this form.

	$y = -5x + 1$
Add $5x$ to both sides.	$y + 5x = -5x + 1 + 5x$

Simplify.	$y + 5x = 1$
Use the Commutative Property to put it in $Ax + By = C$.	$Ax + By = C$ $5x + \ \ y = 1$

By rewriting $y = -5x + 1$ as $5x + y = 1$, we can see that it is a linear equation in two variables because it can be written in the form $Ax + By = C$.

Linear equations in two variables have infinitely many solutions. For every number that is substituted for x, there is a corresponding y value. This pair of values is a **solution to the linear equation** and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement because the value on the left side is equal to the value on the right side.

Note:
Solution to a Linear Equation in Two Variables
An ordered pair (x, y) is a solution to the linear equation $Ax + By = C$, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

Example:
Exercise:

Problem:

Determine which ordered pairs are solutions of the equation $x + 4y = 8$:

ⓐ $(0, 2)$
ⓑ $(2, -4)$

ⓒ $(-4, 3)$

Solution:
Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if the result is a true statement.

ⓐ $(0, 2)$	ⓑ $(2, -4)$	ⓒ $(-4, 3)$
$x = 0, y = 2$ $x + 4y = 8$ $0 + 4 \cdot 2 \stackrel{?}{=} 8$ $0 + 8 \stackrel{?}{=} 8$ $8 = 8$ ✓	$x = 2, y = -4$ $x + 4y = 8$ $2 + 4(-4) \stackrel{?}{=} 8$ $2 + (-16) \stackrel{?}{=} 8$ $-14 \neq 8$	$x = -4, y = 3$ $x + 4y = 8$ $-4 + 4 \cdot 3 \stackrel{?}{=} 8$ $-4 + 12 \stackrel{?}{=} 8$ $8 = 8$ ✓
$(0, 2)$ is a solution.	$(2, -4)$ is not a solution.	$(-4, 3)$ is a solution.

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions to the given equation:
$2x + 3y = 6$

ⓐ $(3, 0)$
ⓑ $(2, 0)$
ⓒ $(6, -2)$

Solution:

ⓐ, ⓒ

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions to the given equation:
$4x - y = 8$

ⓐ $(0, 8)$
ⓑ $(2, 0)$
ⓒ $(1, -4)$

Solution:

ⓑ, ⓒ

Example:
Exercise:

Problem:

Determine which ordered pairs are solutions of the equation.
$y = 5x - 1$:

ⓐ $(0, -1)$
ⓑ $(1, 4)$
ⓒ $(-2, -7)$

Solution:
Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if it results in a true statement.

ⓐ $(0, -1)$	ⓑ $(1, 4)$	ⓒ $(-2, -7)$
$x = 0, y = -1$ $y = 5x - 1$ $-1 \stackrel{?}{=} 5(0) - 1$ $-1 \stackrel{?}{=} 0 - 1$ $-1 = -1$ ✓	$x = 1, y = 4$ $y = 5x - 1$ $4 \stackrel{?}{=} 5(1) - 1$ $4 \stackrel{?}{=} 5 - 1$ $4 = 4$ ✓	$x = -2, y = -7$ $y = 5x - 1$ $-7 \stackrel{?}{=} 5(-2) - 1$ $-7 \stackrel{?}{=} -10 - 1$ $-7 \neq -11$
$(0, -1)$ is a solution.	$(1, 4)$ is a solution.	$(-2, -7)$ is not a solution.

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions of the given equation:
$y = 4x - 3$

ⓐ $(0, 3)$
ⓑ $(1, 1)$
ⓒ $(1, 1)$

Solution:

ⓑ

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions of the given equation:
$y = -2x + 6$

ⓐ $(0, 6)$
ⓑ $(1, 4)$
ⓒ $(-2, -2)$

Solution:

ⓐ, ⓑ

Complete a Table of Solutions to a Linear Equation

In the previous examples, we substituted the x- and y-values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do we find the ordered pairs if they are not given? One way is to choose a value for x and then solve the equation for y. Or, choose a value for y and then solve for x.

We'll start by looking at the solutions to the equation $y = 5x - 1$ we found in [link]. We can summarize this information in a table of solutions.

$y = 5x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$

To find a third solution, we'll let $x = 2$ and solve for y.

	$y = 5x - 1$
Substitute $x = 2$.	$y = 5(2) - 1$

Multiply.	$y = 10 - 1$
Simplify.	$y = 9$

The ordered pair is a solution to $y = 5x - 1$. We will add it to the table.

$y = 5x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$
2	9	$(2, 9)$

We can find more solutions to the equation by substituting any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are an infinite number of solutions for this equation.

Example:
Exercise:

Problem:

Complete the table to find three solutions to the equation $y = 4x - 2$:

$y = 4x - 2$		
x	y	(x, y)
0		
-1		
2		

Solution:
Solution

Substitute $x = 0$, $x = -1$, and $x = 2$ into $y = 4x - 2$.

$x = 0$	$x = -1$	$x = 2$
$y = 4x - 2$	$y = 4x - 2$	$y = 4x - 2$
$y = 4 \cdot 0 - 2$	$y = 4(-1) - 2$	$y = 4 \cdot 2 - 2$
$y = 0 - 2$	$y = -4 - 2$	$y = 8 - 2$
$y = -2$	$y = -6$	$y = 6$
$(0, -2)$	$(-1, -6)$	$(2, 6)$

The results are summarized in the table.

$y = 4x - 2$		
x	y	(x, y)
0	-2	$(0, -2)$
-1	-6	$(-1, -6)$
2	6	$(2, 6)$

Note:
Exercise:

Problem:

Complete the table to find three solutions to the equation: $y = 3x - 1$.

$y = 3x - 1$		
x	y	(x, y)

$y = 3x - 1$		
x	y	(x, y)
0		
-1		
2		

Solution:

$y = 3x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
-1	-4	$(-1, -4)$
2	5	$(2, 5)$

Note:
Exercise:

Problem:

Complete the table to find three solutions to the equation: $y = 6x + 1$

$y = 6x + 1$		
x	y	(x, y)
0		
1		
-2		

Solution:

$y = 6x + 1$		
x	y	(x, y)
0	1	$(0, 1)$
1	7	$(1, 7)$
-2	-11	$(-2, -11)$

Example:
Exercise:

Problem:

Complete the table to find three solutions to the equation $5x - 4y = 20$:

$5x - 4y = 20$		
x	y	(x, y)
0		
	0	
	5	

Solution:
Solution

$x = 0$	$y = 0$	$y = 5$
$5x - 4y = 20$	$5x - 4y = 20$	$5x - 4y = 20$
$5 \cdot 0 - 4y = 20$	$5x - 4 \cdot 0 = 20$	$5x - 4 \cdot 5 = 20$
$0 - 4y = 20$	$5x - 0 = 20$	$5x - 20 = 20$
$-4y = 20$	$5x = 20$	$5x = 40$
$y = -5$	$x = 4$	$x = 8$
$(0, -5)$	$(4, 0)$	$(8, 5)$

The results are summarized in the table.

$5x - 4y = 20$		
x	y	(x, y)
0	-5	$(0, -5)$
4	0	$(4, 0)$
8	5	$(8, 5)$

Note:
Exercise:

Problem:

Complete the table to find three solutions to the equation:
$2x - 5y = 20.$

$2x - 5y = 20$		
x	y	(x, y)
0		
	0	
-5		

Solution:

$2x - 5y = 20$		
x	y	(x, y)
0	-4	$(0, -4)$
10	0	$(10, 0)$
-5	-6	$(-5, -6)$

Note:
Exercise:

Problem:

Complete the table to find three solutions to the equation:
$3x - 4y = 12$.

$3x - 4y = 12$		
x	y	(x, y)
0		
	0	
-4		

Solution:

$3x - 4y = 12$		
x	y	(x, y)
0	-3	$(0, -3)$
4	0	$(4, 0)$

$3x - 4y = 12$		
x	y	(x, y)
-4	-6	$(-4, -6)$

Find Solutions to Linear Equations in Two Variables

To find a solution to a linear equation, we can choose any number we want to substitute into the equation for either x or y. We could choose $1, 100, 1{,}000$, or any other value we want. But it's a good idea to choose a number that's easy to work with. We'll usually choose 0 as one of our values.

Example:
Exercise:

Problem: Find a solution to the equation $3x + 2y = 6$.

Solution:
Solution

Step 1: Choose any value for one of the variables in the equation.		We can substitute any value we want for x or any

		value for y. Let's pick $x = 0$. What is the value of y if $x = 0$?
Step 2: Substitute that value into the equation. Solve for the other variable.	Substitute 0 for x. Simplify. Divide both sides by 2.	$3x + 2y = 6$ $3 \cdot 0 + 2y = 6$ $0 + 2y = 6$ $2y = 6$ $y = 3$
Step 3: Write the solution as an ordered pair.	So, when $x = 0, y = 3$.	This solution is represented by the ordered pair $(0, 3)$.
Step 4: Check.	Substitute $x = 0$, $y = 3$ into the equation $3x + 2y = 6$. Is the result a true equation? Yes!	$3x + 2y = 6$ $3 \cdot 0 + 2 \cdot 3 \stackrel{?}{=} 6$ $0 + 6 \stackrel{?}{=} 6$ $6 = 6$ ✓

Note:
Exercise:

Problem: Find a solution to the equation: $4x + 3y = 12$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find a solution to the equation: $2x + 4y = 8$.

Solution:

Answers will vary.

We said that linear equations in two variables have infinitely many solutions, and we've just found one of them. Let's find some other solutions to the equation $3x + 2y = 6$.

Example:
Exercise:

Problem: Find three more solutions to the equation $3x + 2y = 6$.

Solution:
Solution

To find solutions to $3x + 2y = 6$, choose a value for x or y. Remember, we can choose any value we want for x or y. Here we chose 1 for x, and 0 and -3 for y.

Substitute it into the equation.	$y = 0$ $3x + 2y = 6$ $3x + 2(0) = 6$	$x = 1$ $3x + 2y = 6$ $3(1) + 2y = 6$	$y = -3$ $3x + 2y = 6$ $3x + 2(-3) = 6$
Simplify. Solve.	$3x + 0 = 6$ $3x = 6$	$3 + 2y = 6$ $2y = 3$	$3x - 6 = 6$ $3x = 12$
	$x = 2$	$y = \frac{3}{2}$	$x = 4$
Write the ordered pair.	$(2, 0)$	$(1, \frac{3}{2})$	$(4, -3)$

Check your answers.

$(2, 0)$	$(1, \frac{3}{2})$	$(4, -3)$
$3x + 2y = 6$ $3 \cdot 2 + 2 \cdot 0 \stackrel{?}{=} 6$ $6 + 0 \stackrel{?}{=} 6$ $6 = 6 \checkmark$	$3x + 2y = 6$ $3 \cdot 1 + 2 \cdot \frac{3}{2} \stackrel{?}{=} 6$ $3 + 3 \stackrel{?}{=} 6$ $6 = 6 \checkmark$	$3x + 2y = 6$ $3 \cdot 4 + 2(-3) \stackrel{?}{=} 6$ $12 + (-6) \stackrel{?}{=} 6$ $6 = 6 \checkmark$

So $(2, 0)$, $\left(1, \frac{3}{2}\right)$ and $(4, -3)$ are all solutions to the equation $3x + 2y = 6$. In the previous example, we found that $(0, 3)$ is a solution, too. We can list these solutions in a table.

$3x + 2y = 6$		
x	y	(x, y)
0	3	$(0, 3)$
2	0	$(2, 0)$
1	$\frac{3}{2}$	$(1, \frac{3}{2})$
4	-3	$(4, -3)$

Note:
Exercise:

Problem: Find three solutions to the equation: $2x + 3y = 6$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find three solutions to the equation: $4x + 2y = 8$.

Solution:

Answers will vary.

Let's find some solutions to another equation now.

Example:
Exercise:

Problem: Find three solutions to the equation $x - 4y = 8$.

Solution:
Solution

	$x - 4y = 8$	$x - 4y = 8$	$x - 4y = 8$
Choose a value for x or y.	$x = 0$	$y = 0$	$y = 3$
Substitute it into the equation.	$0 - 4y = 8$	$x - 4 \cdot 0 = 8$	$x - 4 \cdot 3 = 8$

Solve.	$-4y = 8$ $y = -2$	$x - 0 = 8$ $x = 8$	$x - 12 = 8$ $x = 20$
Write the ordered pair.	$(0, -2)$	$(8, 0)$	$(20, 3)$

So $(0, -2)$, $(8, 0)$, and $(20, 3)$ are three solutions to the equation $x - 4y = 8$.

$x - 4y = 8$		
x	y	(x, y)
0	-2	$(0, -2)$
8	0	$(8, 0)$
20	3	$(20, 3)$

Remember, there are an infinite number of solutions to each linear equation. Any point you find is a solution if it makes the equation true.

Note:
Exercise:

Problem: Find three solutions to the equation: $4x + y = 8$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find three solutions to the equation: $x + 5y = 10$.

Solution:

Answers will vary.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Plotting Points
- Identifying Quadrants
- Verifying Solution to Linear Equation

Key Concepts

- **Sign Patterns of the Quadrants**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$

- **Coordinates of Zero**
    - Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
    - Points with a x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
    - The point $(0, 0)$ is called the origin. It is the point where the x-axis and y-axis intersect.

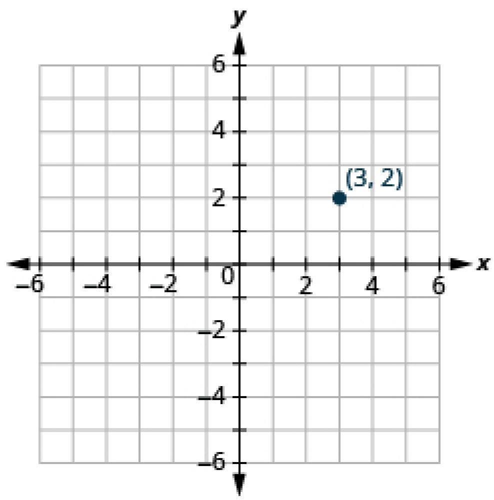
Practice Makes Perfect

Plot Points on a Rectangular Coordinate System

In the following exercises, plot each point on a coordinate grid.
Exercise:

Problem: $(3,2)$

Solution:



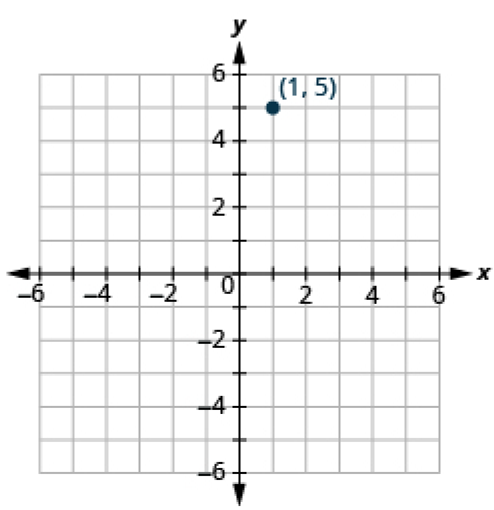
Exercise:

Problem: $(4, 1)$

Exercise:

Problem: $(1, 5)$

Solution:



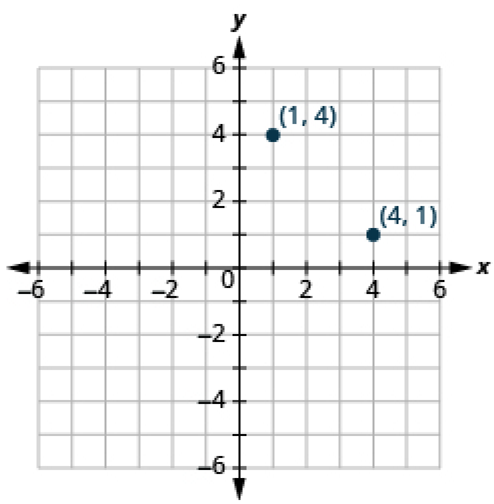
Exercise:

Problem: $(3, 4)$

Exercise:

Problem: $(4, 1), (1, 4)$

Solution:



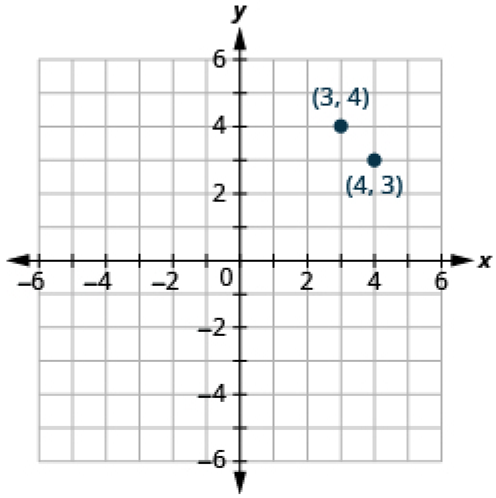
Exercise:

Problem: $(3, 2), (2, 3)$

Exercise:

Problem: $(3, 4), (4, 3)$

Solution:



In the following exercises, plot each point on a coordinate grid and identify the quadrant in which the point is located.

Exercise:

Problem:

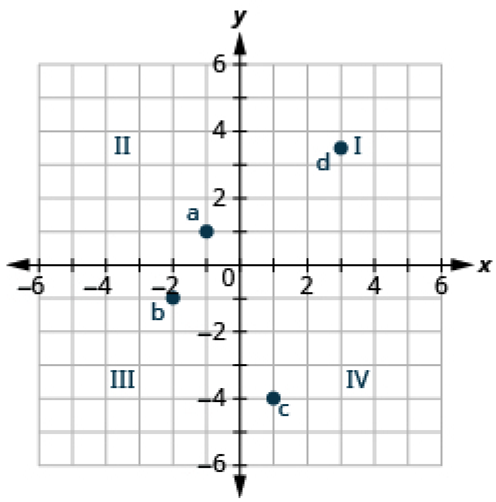
ⓐ $(-4, 2)$
ⓑ $(-1, -2)$
ⓒ $(3, -5)$
ⓓ $\left(2, \frac{5}{2}\right)$

Exercise:

Problem:

ⓐ $(-2, -3)$
ⓑ $(3, -3)$
ⓒ $(-4, 1)$
ⓓ $\left(1, \frac{3}{2}\right)$

Solution:



Exercise:

Problem:

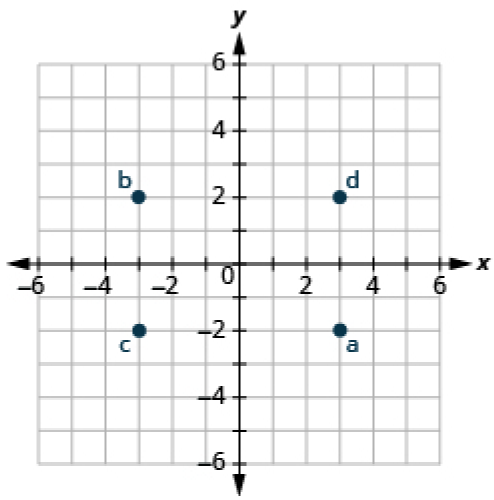
ⓐ $(-1, 1)$
ⓑ $(-2, -1)$
ⓒ $(1, -4)$
ⓓ $\left(3, \frac{7}{2}\right)$

Exercise:

Problem:

ⓐ $(3, -2)$
ⓑ $(-3, 2)$
ⓒ $(-3, -2)$
ⓓ $(3, 2)$

Solution:



Exercise:

Problem:

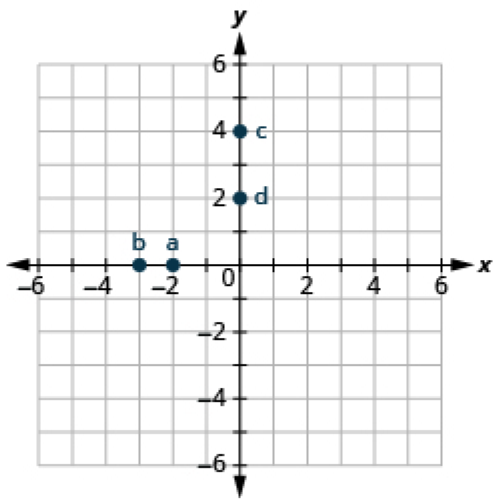
ⓐ $(4, -1)$
ⓑ $(-4, 1)$
ⓒ $(-4, -1)$
ⓓ $(4, 1)$

Exercise:

Problem:

- ⓐ $(-2, 0)$
- ⓑ $(-3, 0)$
- ⓒ $(0, 4)$
- ⓓ $(0, 2)$

Solution:

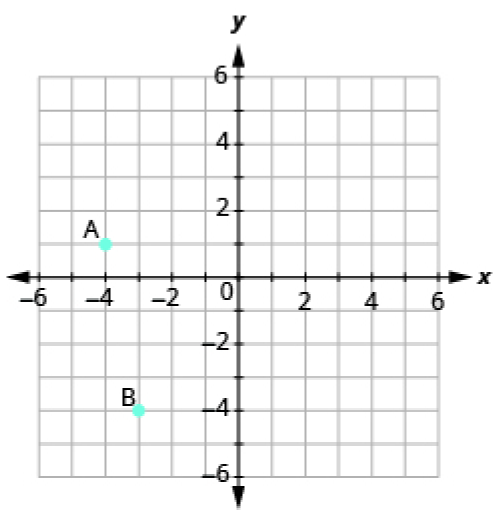


Identify Points on a Graph

In the following exercises, name the ordered pair of each point shown.

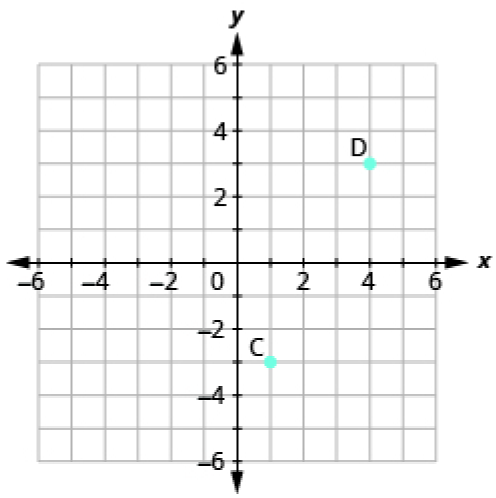
Exercise:

Problem:



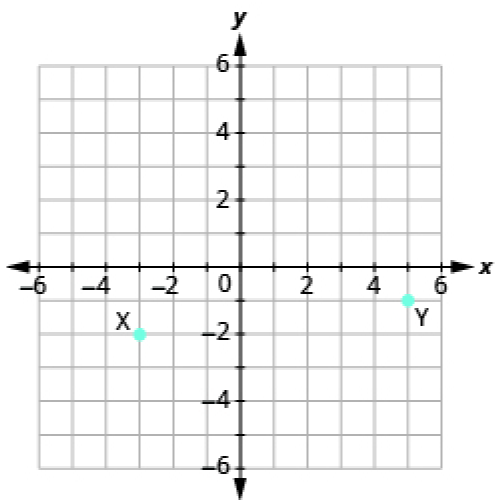
Exercise:

Problem:



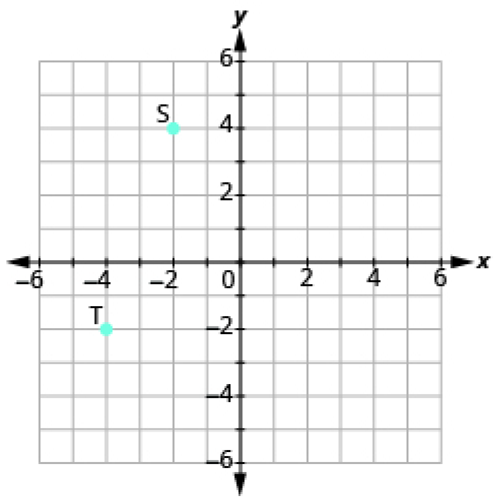
Exercise:

Problem:



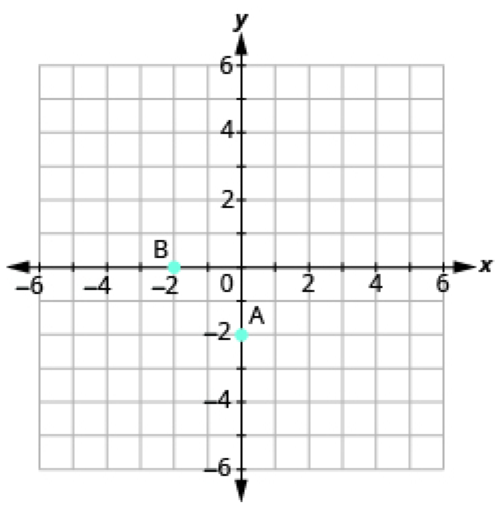
Exercise:

Problem:



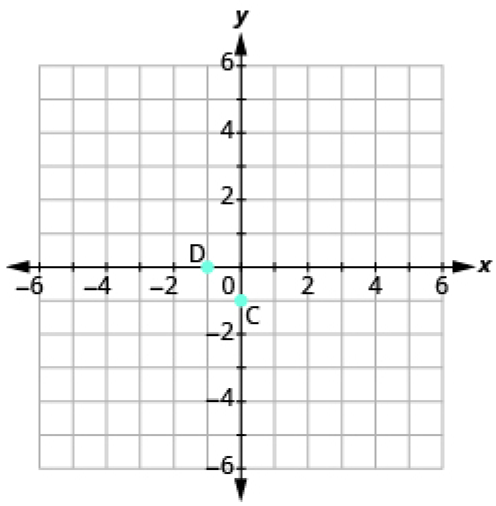
Exercise:

Problem:



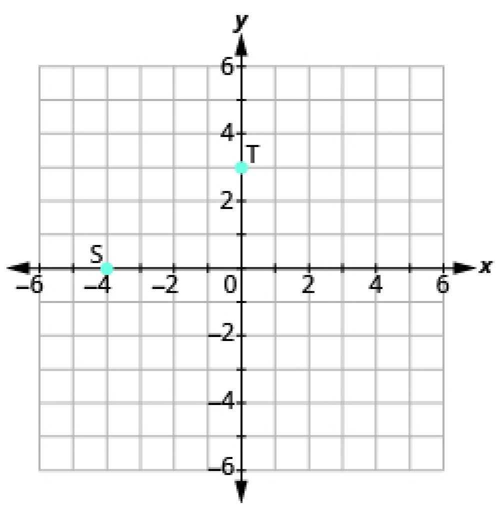
Exercise:

Problem:



Exercise:

Problem:



Verify Solutions to an Equation in Two Variables

In the following exercises, determine which ordered pairs are solutions to the given equation.

Exercise:

Problem: $2x + y = 6$

ⓐ $(1, 4)$
ⓑ $(3, 0)$
ⓒ $(2, 3)$

Solution:

ⓐ, ⓑ

Exercise:

Problem: $x + 3y = 9$

ⓐ $(0, 3)$
ⓑ $(6, 1)$
ⓒ $(-3, -3)$

Exercise:

Problem: $4x - 2y = 8$

ⓐ $(3, 2)$
ⓑ $(1, 4)$
ⓒ $(0, -4)$

Solution:

ⓐ, ⓒ

Exercise:

Problem: $3x - 2y = 12$

ⓐ $(4, 0)$
ⓑ $(2, -3)$
ⓒ $(1, 6)$

Exercise:

Problem: $y = 4x + 3$

ⓐ $(4, 3)$
ⓑ $(-1, -1)$
ⓒ $(\frac{1}{2}, 5)$

Solution:

ⓑ, ⓒ

Exercise:

Problem: $y = 2x - 5$

ⓐ $(0, -5)$
ⓑ $(2, 1)$
ⓒ $(\frac{1}{2}, -4)$

Exercise:

Problem: $y = \frac{1}{2}x - 1$

ⓐ $(2, 0)$
ⓑ $(-6, -4)$
ⓒ $(-4, -1)$

Solution:

ⓐ, ⓑ

Exercise:

Problem: $y = \frac{1}{3}x + 1$

ⓐ $(-3, 0)$
ⓑ $(9, 4)$
ⓒ $(-6, -1)$

Find Solutions to Linear Equations in Two Variables

In the following exercises, complete the table to find solutions to each linear equation.

Exercise:

Problem: $y = 2x - 4$

x	y	(x, y)
-1		
0		
2		

Solution:

x	y	(x, y)
-1	-6	$(-1, -6)$
0	-4	$(0, -4)$
2	0	$(2, 0)$

Exercise:

Problem: $y = 3x - 1$

x	y	(x, y)
-1		
0		
2		

Exercise:

Problem: $y = -x + 5$

x	y	(x, y)
-2		
0		
3		

Solution:

x	y	(x, y)
-2	7	$(-2, 7)$
0	5	$(0, 5)$
3	2	$(3, 2)$

Exercise:

Problem: $y = \frac{1}{3}x + 1$

x	y	(x, y)

x	y	(x, y)
0		
3		
6		

Exercise:

Problem: $y = -\frac{3}{2}x - 2$

x	y	(x, y)
-2		
0		
2		

Solution:

x	y	(x, y)

x	y	(x, y)
-2	1	$(-2, 1)$
0	-2	$(0, -2)$
2	-5	$(2, -5)$

Exercise:

Problem: $x + 2y = 8$

x	y	(x, y)
0		
4		
	0	

Everyday Math

Exercise:

Problem:

Weight of a baby Mackenzie recorded her baby's weight every two months. The baby's age, in months, and weight, in pounds, are listed in the table, and shown as an ordered pair in the third column.

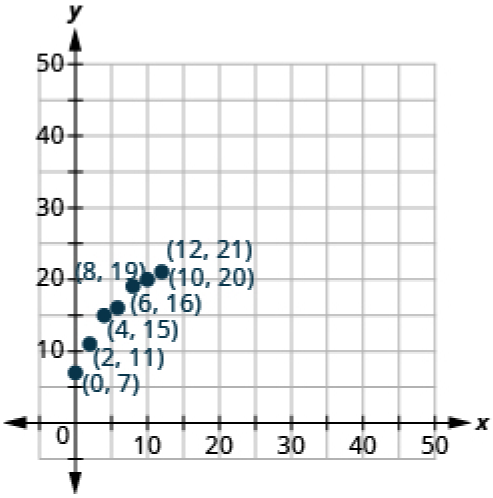
ⓐ Plot the points on a coordinate grid.

Age	Weight	(x, y)
0	7	$(0, 7)$
2	11	$(2, 11)$
4	15	$(4, 15)$
6	16	$(6, 16)$
8	19	$(8, 19)$
10	20	$(10, 20)$
12	21	$(12, 21)$

ⓑ Why is only Quadrant I needed?

Solution:

ⓐ



ⓑ Age and weight are only positive.

Exercise:

Problem:

Weight of a child Latresha recorded her son’s height and weight every year. His height, in inches, and weight, in pounds, are listed in the table, and shown as an ordered pair in the third column.

ⓐ Plot the points on a coordinate grid.

Height x	Weight y	(x, y)
28	22	$(28, 22)$
31	27	$(31, 27)$
33	33	$(33, 33)$

37	35	$(37, 35)$
40	41	$(40, 41)$
42	45	$(42, 45)$

ⓑ Why is only Quadrant I needed?

Writing Exercises

Exercise:

Problem:

Have you ever used a map with a rectangular coordinate system? Describe the map and how you used it.

Solution:

Answers may vary.

Exercise:

Problem:

How do you determine if an ordered pair is a solution to a given equation?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
plot points on a rectangular coordinate system.			
identify points on a graph.			
verify solutions to an equation in two variables.			
complete a table of solutions to a linear equation.			
find solutions to linear equations in two variables.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

linear equation
: An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

ordered pair

An ordered pair (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate.
Equation:

$$\underset{x\text{-coordinate},y\text{-coordinate}}{(x, y)}$$

origin
The point $(0, 0)$ is called the origin. It is the point where the the point where the x-axis and y-axis intersect.

quadrants
The x-axis and y-axis divide a rectangular coordinate system into four areas, called quadrants.

solution to a linear equation in two variables
An ordered pair (x, y) is a solution to the linear equation $Ax + By = C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

x-axis
The *x*-axis is the horizontal axis in a rectangular coordinate system.

y-axis
The *y*-axis is the vertical axis on a rectangular coordinate system.

Review of Interval Notation
An introductory explanation on how to use interval notation (versus set notation)

Interval notation can be used to describe an interval on a number line using mathematical symbols.

An interval notation uses one of the following symbols along with the end points of the given interval:

- Open parentheses ( )
- Closed parentheses [ ]
- Infinity ∞
- Negative Infinity $-\infty$
- Union Sign $\cup$ (Note: $\cup$ symbol can be replaced by the word "OR")

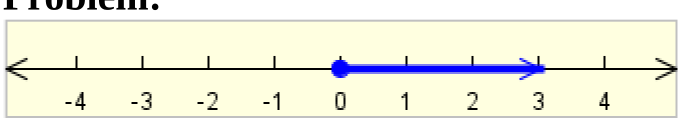
Interval notation uses the end points of a given interval along with either square brackets "[or]" or regular parenthesis "(or)". A square bracket is used to indicate that the interval includes the endpoint whereas a regular parenthesis indicates that the endpoint is excluded from the interval. Since ∞ is not a number, we use (or) to describe intervals that extend to either positive or negative ∞. If the interval goes on forever to the left, the interval notation will start with ($-\infty$. If the interval goes on forever at the right, interval notation will end with ∞).

Whenever there is a break in the interval, write the interval up to the point. Then write another interval for the section of the graph after that part. Put a union sign between each interval to "join" them together.

Now for some practice so you can see if any of this makes sense.

Write the following using interval notation:
Exercise:

Problem:

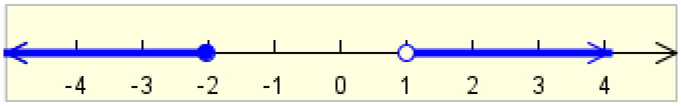


Solution:

$[0, \infty)$

Exercise:

Problem:

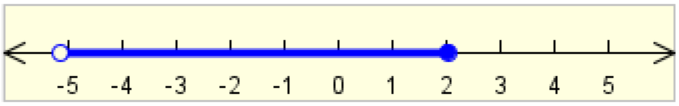


Solution:

$(-\infty, -2] \cup (1, \infty)$

Exercise:

Problem:

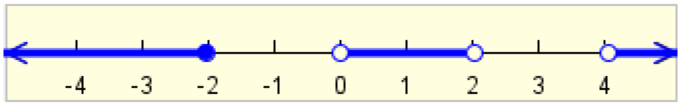


Solution:

$(-5, 2]$

Exercise:

Problem:

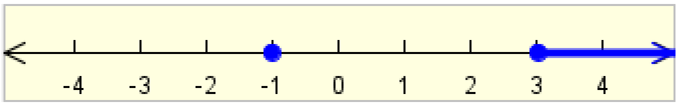


Solution:

$(-\infty, -2] \cup (0, 2) \cup (4, \infty)$

Exercise:

Problem:

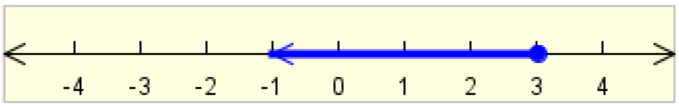


Solution:

$[-1] \cup [3, \infty)$

Exercise:

Problem:



Solution:

$(-\infty, 3]$

Interval Notation - Web Resources
Interval Notation

Interval Notation – Web Resources

Intervals @mathisfun.com

Intervals and Interval Notation @KhanAcademy

Using Interval Notation @LumenLearning

Additional Resource

https://www.youtube.com/embed/pVOHVj2aFsg

Understand Interval Notation

Linear Inequalities and Interval Notation
By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve inequalities using the Subtraction and Addition Properties of inequality
- Solve inequalities using the Division and Multiplication Properties of inequality
- Solve inequalities that require simplification
- Translate to an inequality and solve

Note:
Before you get started, take this readiness quiz.

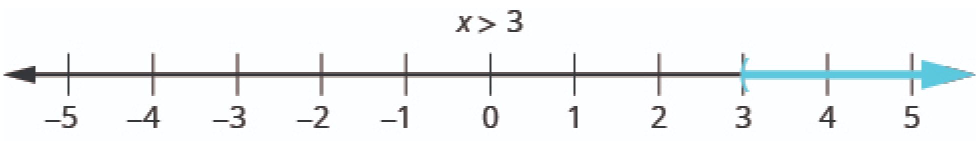
1. Translate from algebra to English: $15 > x$.
   If you missed this problem, review [link].
2. Solve: $n - 9 = -42$.
   If you missed this problem, review [link].
3. Solve: $-5p = -23$.
   If you missed this problem, review [link].
4. Solve: $3a - 12 = 7a - 20$.
   If you missed this problem, review [link].

Graph Inequalities on the Number Line

Do you remember what it means for a number to be a solution to an equation? A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

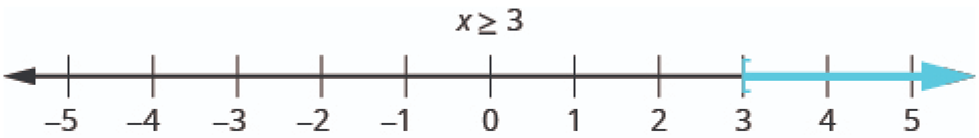
What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, '*x* could be 4'? That's correct, but *x* could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put an open parenthesis at 3. The graph of $x > 3$ is shown in [link]. Please note that the following convention is used: light blue arrows point in the positive direction and dark blue arrows point in the negative direction.



The inequality $x > 3$ is graphed on this number line.

The graph of the inequality $x \geq 3$ is very much like the graph of $x > 3$, but now we need to show that 3 is a solution, too. We do that by putting a bracket at $x = 3$, as shown in [link].



The inequality $x \geq 3$ is graphed on this number line.

Notice that the open parentheses symbol, (, shows that the endpoint of the inequality is not included. The open bracket symbol, [, shows that the endpoint is included.

Example:

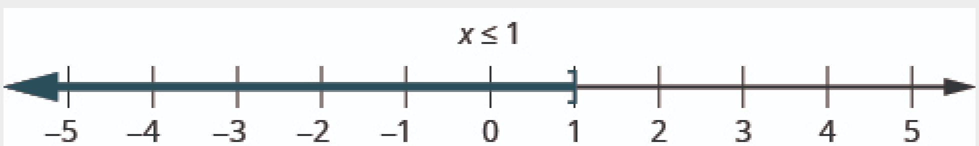
Exercise:

Problem: Graph on the number line:

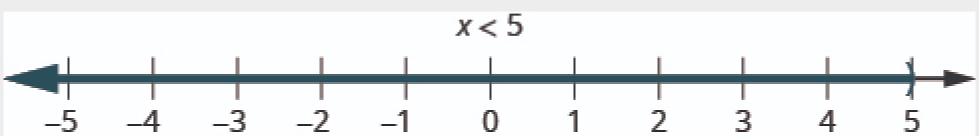
ⓐ $x \leq 1$ ⓑ $x < 5$ ⓒ $x > -1$

Solution:
Solution

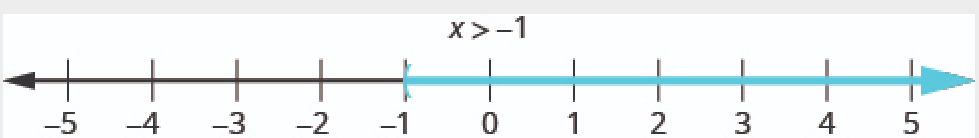
ⓐ $x \leq 1$
This means all numbers less than or equal to 1. We shade in all the numbers on the number line to the left of 1 and put a bracket at $x = 1$ to show that it is included.



ⓑ $x < 5$
This means all numbers less than 5, but not including 5. We shade in all the numbers on the number line to the left of 5 and put a parenthesis at $x = 5$ to show it is not included.



ⓒ $x > -1$
This means all numbers greater than -1, but not including -1. We shade in all the numbers on the number line to the right of -1, then put a parenthesis at $x = -1$ to show it is not included.

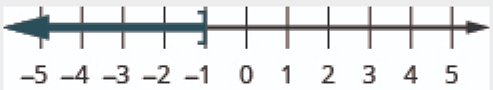


Note:
Exercise:

Problem: Graph on the number line: ⓐ $x \leq -1$ ⓑ $x > 2$ ⓒ $x < 3$

Solution:

ⓐ



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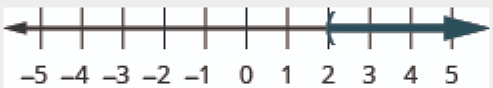

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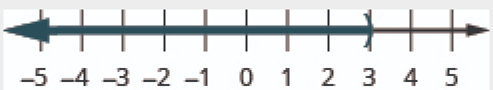


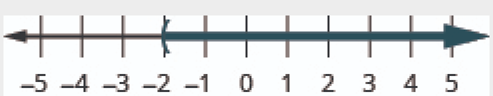
Note:
Exercise:

Problem:

Graph on the number line: ⓐ $x > -2$ ⓑ $x < -3$ ⓒ $x \geq -1$

Solution:

ⓐ



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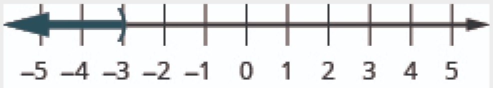

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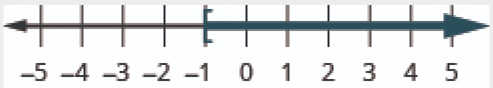
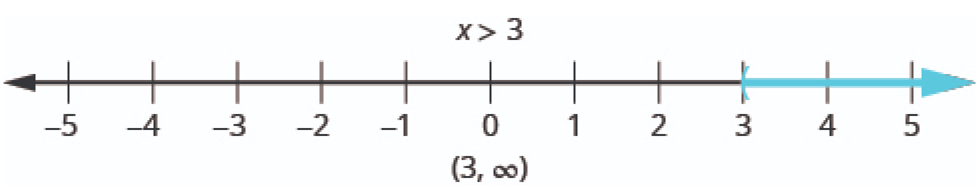
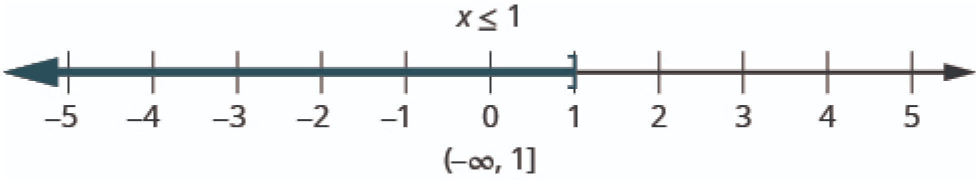


We can also represent inequalities using *interval notation.* As we saw above, the inequality $x > 3$ means all numbers greater than 3. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as ‘infinity’. It is not an actual number. [link] shows both the number line and the interval notation.



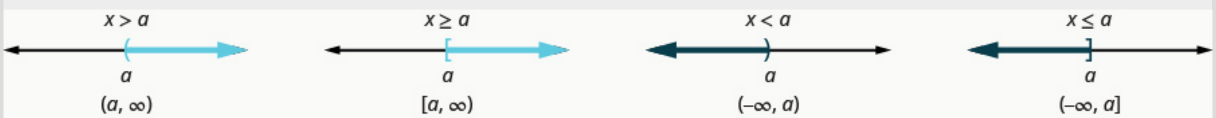
The inequality $x > 3$ is graphed on this number line and written in interval notation.

The inequality $x \leq 1$ means all numbers less than or equal to 1. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as ‘negative infinity’. [link] shows both the number line and interval notation.



The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Note:
Inequalities, Number Lines, and Interval Notation



Did you notice how the parenthesis or bracket in the interval notation matches the symbol at the endpoint of the arrow? These relationships are shown in [link].

The notation for inequalities on a number line and in interval notation use similar symbols to express the endpoints of intervals.

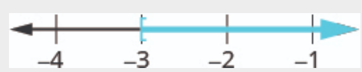
Example:
Exercise:

Problem: Graph on the number line and write in interval notation.

ⓐ $x \geq -3$ ⓑ $x < 2.5$ ⓒ $x \leq -\frac{3}{5}$

Solution:
Solution

ⓐ

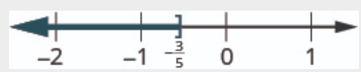
	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

	$x < 2.5$

Shade to the left of 2.5, and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

Ⓒ

	$x \le -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

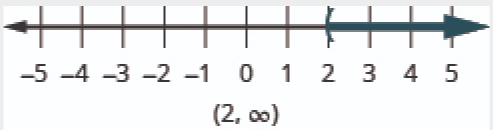
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

ⓐ $x > 2$
ⓑ $x \leq -1.5$
ⓒ $x \geq \frac{3}{4}$

Solution:

ⓐ



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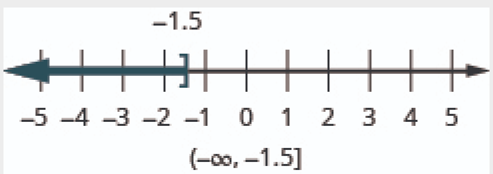

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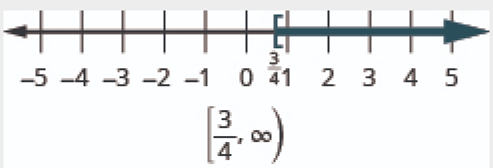


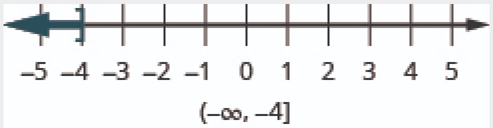
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

ⓐ $x \leq -4$
ⓑ $x \geq 0.5$
ⓒ $x < -\frac{2}{3}$

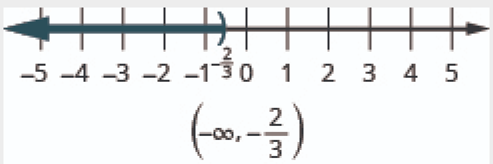
Solution:

ⓐ



ⓑ

ⓒ



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like ‘more than’ and ‘less than’. But others are not as obvious.

Think about the phrase ‘at least’ – what does it mean to be ‘at least 21 years old’? It means 21 or more. The phrase ‘at least’ is the same as ‘greater than or equal to’.

[link] shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Example:
Exercise:

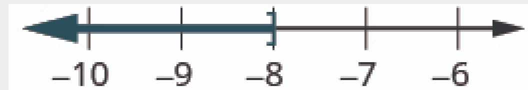
Problem:

Rewrite as an inequality. Then write the solution in interval notation and graph on the number line.

c is no more than eight.

Solution:
Solution

Translate.	$c \leq 8$

Write in interval notation.	$(-\infty, -8]$
Graph on the number line.	

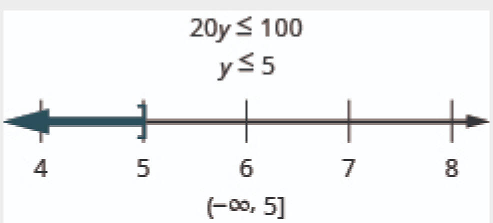
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twenty times *y* is at most 100

Solution:



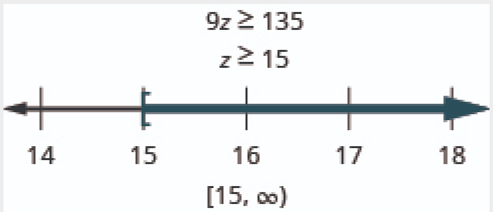
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nine times *z* is no less than 135

Solution:



Practice Makes Perfect

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.

Exercise:

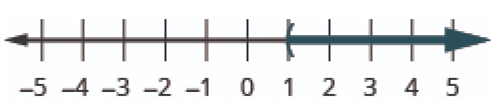
Problem: ⓐ $x \leq 2$ ⓑ $x > -1$ ⓒ $x < 0$

Exercise:

Problem: ⓐ $x > 1$ ⓑ $x < -2$ ⓒ $x \geq -3$

Solution:

ⓐ



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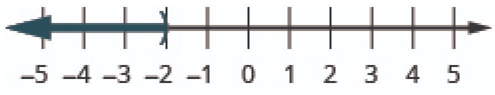

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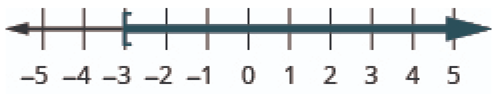


Exercise:

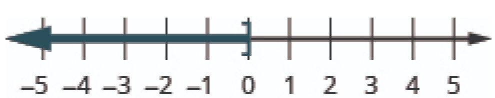
Problem:
ⓐ $x \geq -3$
ⓑ $x < 4$
ⓒ $x \leq -2$

Exercise:

Problem:
ⓐ $x \leq 0$
ⓑ $x > -4$
ⓒ $x \geq -1$

Solution:

ⓐ



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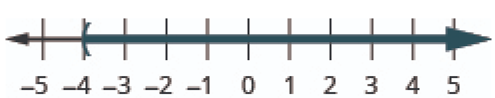

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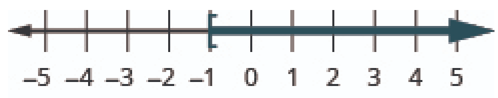


In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

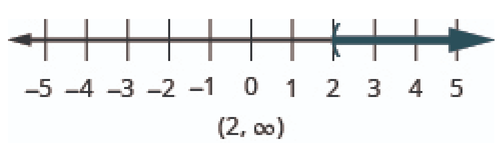
Problem: ⓐ $x < -1$
ⓑ $x \geq -2.5$
ⓒ $x \leq \frac{5}{4}$

Exercise:

Problem: ⓐ $x > 2$
ⓑ $x \leq -1.5$
ⓒ $x \geq \frac{5}{3}$

Solution:

ⓐ



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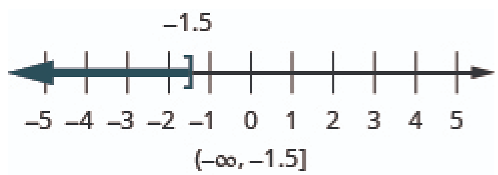

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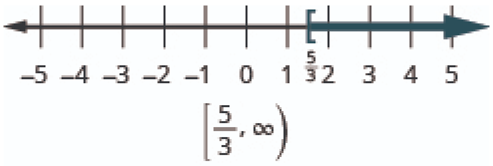


Exercise:

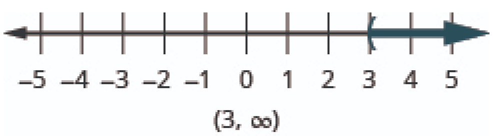
Problem:
ⓐ $x < -2$
ⓑ $x \geq -3.5$
ⓒ $x \leq \frac{2}{3}$

Exercise:

Problem:
ⓐ $x > 3$
ⓑ $x \leq -0.5$
ⓒ $x \geq \frac{1}{3}$

Solution:

ⓐ



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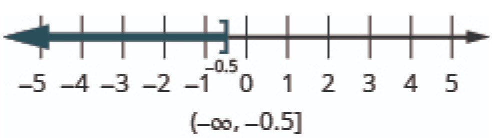

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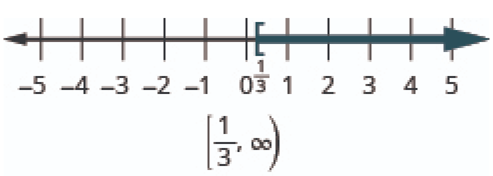


Exercise:

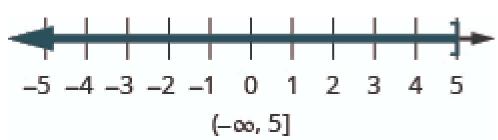
Problem: ⓐ $x \geq -4$
ⓑ $x < 2.5$
ⓒ $x > -\frac{3}{2}$

Exercise:

Problem: ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
ⓒ $x < -\frac{7}{3}$

Solution:

ⓐ



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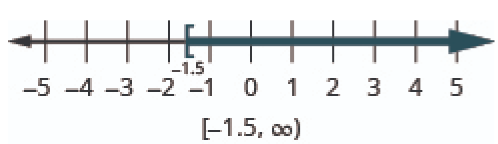

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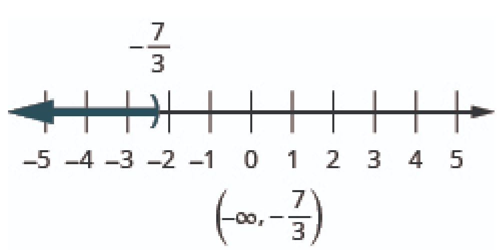


Translate to an Inequality and Solve

In the following exercises, translate and solve .Then write the solution in interval notation and graph on the number line.

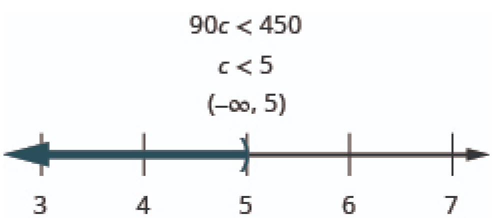
Exercise:

Problem: Fourteen times *d* is greater than 56.

Exercise:

Problem: Ninety times *c* is less than 450.

Solution:



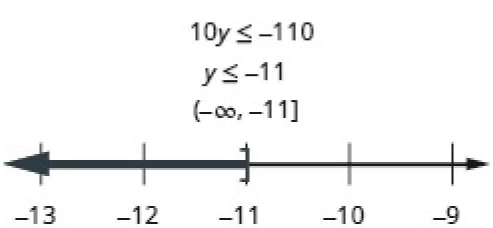
Exercise:

Problem: Eight times *z* is smaller than -40.

Exercise:

Problem: Ten times *y* is at most -110.

Solution:



Everyday Math

Exercise:

Problem:

Safety A child’s height, *h*, must be at least 57 inches for the child to safely ride in the front seat of a car. Write this as an inequality.

Exercise:

Problem:

Fighter pilots The maximum height, *h*, of a fighter pilot is 77 inches. Write this as an inequality.

Solution:

$h \le 77$

Exercise:

Problem:

Elevators The total weight, *w*, of an elevator’s passengers can be no more than 1,200 pounds. Write this as an inequality.

Exercise:

Problem:

Shopping The number of items, *n*, a shopper can have in the express check-out lane is at most 8. Write this as an inequality.

Solution:

$n \le 8$

Writing Exercises

Exercise:

Problem: Give an example from your life using the phrase ‘at least’.

Exercise:

Problem: Give an example from your life using the phrase ‘at most’.

Solution:

Answers will vary.

The Order of Operations
A review of the order of operations, using whole numbers, in preparation to working with integers and real numbers.

Grouping Symbols

Grouping symbols are used to indicate that a particular collection of numbers and meaningful operations are to be grouped together and considered as one number. They help to make clear which expressions are to be kept together and separate from other expressions. The grouping symbols commonly used in mathematics are:

Parentheses: ( )

Brackets: [ ]

Braces: { }

Fraction Bar: —

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

Example:
$8(14-8);\qquad 21-3[2+4(9-8)];\qquad 24 \div \{13-2[1(6-5)+4]\}$

In a computation in which more than one operation is involved, grouping symbols indicate which operation to perform first. If possible, we perform operations inside grouping symbols first.

For example:

$(5 \cdot 5) + 20 = 45$ since we must multiply 5 times 5 to obtain 25 first and then add the 20.

whereas:

$5 \cdot (5 + 20) = 125$ since we must add 5 and 20 first to obtain 25 and then multiply that result times 5.

Grouping Symbols Examples

If possible, determine the value of each of the following.

Example 1

$9 + (3 \cdot 8)$

Since 3 and 8 are within parentheses, they are to be combined first:

$= 9 + 24$

Then add the terms:

$= 33$

Thus, $9 + (3 \cdot 8) = 33$.

Example 2

$(10 \div 0) \cdot 6$

Since $(10 \div 0)$ is undefined, we attach no value to it. We write, "undefined."

Grouping Symbols Practice Exercises

If possible, determine the value of each of the following.

Note:
Try It
Exercise:

Problem: $16 – (3 \cdot 2)$

Solution:

10

Note:
Try It
Exercise:

Problem: $5 + (7 \cdot 9)$

Solution:

68

Note:
Try It
Exercise:

Problem: $(4 + 8) \cdot 2$

Solution:

24

Note:
Try It
Exercise:

Problem: $28 \div (18 - 11)$

Solution:

4

Note:
Try It
Exercise:

Problem: $(33 \div 3) - 11$

Solution:

0

Note:
Try It
Exercise:

Problem: $4 + (0 \div 0)$

Solution:

undefined

Multiple Grouping Symbols

When a set of grouping symbols occurs inside another set of grouping symbols, we perform the operations within the innermost set first.

When simplifying expressions with multiple operations, you must perform all multiplications and divisions first (in order as you move from left to right) followed by all additions and subtractions (in order as you move from left to right).

Multiple Grouping Symbol Examples

Determine the value of each of the following.

Example:
$2 + (8 \cdot 3) - (5 + 6)$
Combine 8 and 3 first, then combine 5 and 6.
$= 2 + 24 - 11$
Now combine left to right.
$= 26 - 11$
$= 15$

Example:
$10 + [30 - (2 \cdot 9)]$
Combine 2 and 9 since they occur in the innermost set of parentheses.
$= 10 + [30 - 18]$
Now combine 30 and 18.
$= 10 + 12$
$= 22$

Multiple Grouping Symbol Practice Exercises

Determine the value of each of the following:

Note:
Try It
Exercise:

Problem: $(17 + 8) + (9 + 20)$

Solution:

54

Note:
Try It
Exercise:

Problem: $(55 - 6) + (13 \cdot 2)$

Solution:

75

Note:
Try It
Exercise:

Problem: $23 + (12 \div 4) + (11 \cdot 2)$

Solution:

48

Note:
Try It
Exercise:

Problem: $86 + [14 + (10 - 8)]$

Solution:

102

Note:
Try It
Exercise:

Problem: $31 + (9 + [1 + (35 - 2)])$

Solution:

74

Order of Operations (PEMDAS)

To **simplify an expression**, we need to perform all operations in the given expression. Sometimes there are no grouping symbols indicating which operations to perform first. For example, suppose we wish to find the value of the expression:

Equation:

$$3 + 5 \cdot 2$$

If we simplify this expression, what do we get?

a. Some students say 16 because we add 3 and 5, then multiply this sum by 2 to get 16.

Equation:

$$\begin{array}{ll} & 3 + 5 \cdot 2 \\ \text{Since } 3 + 5 \text{ gives } 8. & 8 \cdot 2 \\ \text{And, } 8 \cdot 2 \text{ is } 16. & 16 \end{array}$$

b. Others say the answer is 13 since we multiply 5 and 2, then add 3 to this product to arrive at 13.

Equation:

$$\begin{array}{ll} & 3 + 5 \cdot 2 \\ \text{Since } 5 \cdot 2 \text{ is } 10. & 3 + 10 \\ \text{And, } 3 + 10 \text{ equals } 13. & 13 \end{array}$$

We now have two values for the same expression!

In order to avoid this confusion, we need a set of rules so that everyone obtains the same result for a given expression.expression. So mathematicians early on established some guidelines that are called the Order of Operations.

The universally agreed-upon **order of operations**, often referred to as PEMDAS, for evaluating a mathematical expression is as follows:

P: Parentheses (and other grouping symbols) from the inside out.
Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

By parentheses we mean anything that acts as a grouping symbol, including anything inside symbols such as [ ], { }, | |, and $\sqrt{\ }$.

Any expression in the numerator or denominator of a fraction or in an exponent is also considered grouped, and should be simplified before carrying out further operations.

If there are nested parentheses (parentheses inside parentheses), you work from the innermost parentheses outward.

E: Exponents
Simplify all expressions with exponents.

MD: Multiplications and Divisions
Perform all multiplication and division in order from left to right. These operations have EQUAL priority.

AS: Additions and Subtractions
Perform all addition and subtraction in order from left to right. These operations have EQUAL priority.

Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase: “Please Excuse My Dear Aunt Sally.”
Equation:

Parentheses	**P**lease
Exponents	**E**xcuse
Multiplication **D**ivision	**M**y **D**ear
Addition **S**ubtraction	**A**unt **S**ally

It’s good that “**M**y **D**ear” goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, “**A**unt **S**ally” goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

For example, given: $3 + 15 \div 3 + 5 \times 2^{2+3}$

The exponent is an implied grouping, so the 2+3 must be evaluated first:

$= 3 + 15 \div 3 + 5 \times 2^5$

Now the exponent is carried out:

$= 3 +15 \div 3 + 5 \times 32$

Then the multiplication and division, left to right using $15 \div 3 = 5$ and $5 \times 32 = 160$:

$= 3 + 5 + 160$

Finally, the addition, left to right:

$= 168$

Examples, Order of Operations

Determine the value of each of the following.

Example:
$21 + 3 \cdot 12.$

Follow order of operations: PEMDAS.
Are there any Parenthesis or grouping symbols? Exponents? Multiplications and Divisions? Additions or Subtractions?
Since Multiplication is the operation with the highest order here, we multiply first:
= $21 + 36$
Add.
= 57

Example:
$(15 - 8) + 5(6 + 4)$.
Simplify inside parentheses first.
= $7 + 5 \cdot 10$
Multiply.
= $7 + 50$
Add.
= 57

Example:
$63 - (4 + 6 \cdot 3) + 76 - 4$.
Simplify first within the parentheses by multiplying, then adding:
= $63 - (4 + 18) + 76 - 4$
= $63 - 22 + 76 - 4$
Now perform the additions and subtractions, moving left to right:
= $41 + 76 - 4$
= $117 - 4$

= 113.

Example:
$7 \cdot 6 - 4^2 + 1^5$
Evaluate the exponents.
= $7 \cdot 6 - 16 + 1$
Multiply 7 · 6:
= $42 - 16 + 1$
Perform additions and subtractions from left to right. Subtraction is first, so subtract 16 from 42:
= 26 + 1
Add 26 and 1:
= 27.

Example:
$6 \cdot (3^2 + 2^2) + 4^2$
Evaluate the exponents in the parentheses:
= $6 \cdot (9 + 4) + 4^2$
Add 9 and 4 in the parentheses:
= $6 \cdot (13) + 4^2$
Evaluate the exponential form 4^2:
= $6 \cdot (13) + 16$
Multiply 6 and 13:
= $78 + 16$
Add 78 and 16:
= 94

Example:

$\frac{6^2+2^2}{4^2+6\cdot 2^2} + \frac{1^3+8^2}{10^2-19\cdot 5}$.

Recall that the fraction bar is a grouping symbol. The fraction $\frac{6^2+2^2}{4^2+6\cdot 2^2}$ is equivalent to $\left(6^2+2^2\right) \div \left(4^2+6\cdot 2^2\right)$

Evaluate all exponents.

$= \frac{36+4}{16+6\cdot 4} + \frac{1+64}{100-19\cdot 5}$

Perform all multiplications.

$= \frac{36+4}{16+24} + \frac{1+64}{100-95}$

Perform all additions and subtractions that are above or below the fraction bars.

$= \frac{40}{40} + \frac{65}{5}$

Perform all divisions, denoted by the fraction bars.

= 1+13

Add.

= 14

Order of Operations, Practice Exercises

Determine the value of the following:

Note:

Try It

Exercise:

Problem: 2{8 + (32 – 7)}

Solution:

66

Note:
Try It
Exercise:

Problem: $(34 + 18 – 2 \cdot 3) + 11$

Solution:

57

Note:
Try It
Exercise:

Problem: $8(10) + 4(2 + 3) – (20 + 3 \cdot 15 + 40 – 5)$

Solution:

0

Note:
Try It
Exercise:

Problem: $5 \cdot 8 + 42 – 22$

Solution:

60

Note:
Try It
Exercise:

Problem: $4(6^2 - 3^3) \div (4^2 - 4)$

Solution:

3

Note:
Try It
Exercise:

Problem: $\{6 - [24 \div (4 \cdot 2)]\}^3$

Solution:

27

Note:
Try It
Exercise:

Problem: $(8 + 9 \cdot 3) \div 7 + 5 \cdot (8 \div 4 + 7 + 3 \cdot 5)$

Solution:

125

Homework

For the following problems, find each value.

Exercise:

Problem: $1 - 5(8 - 8)$

Solution:

1

Exercise:

Problem: $37 - 1 \cdot 6^2$

Exercise:

Problem: $98 \div 2 \div 7^2$

Solution:

1

Exercise:

Problem: $(4^2 - 2 \cdot 4) - 2^3$

Exercise:

Problem: $61 - 22 + 4[3 \cdot 10 + 11]$

Solution:

203

Exercise:

Problem: $121 - 4 \cdot [4 \cdot 5 - 12] + \frac{16}{2}$

Exercise:

Problem: $2^2 \cdot 3 + 2^3(6 - 2) - \left(3 + 17\right) + 11(6)$

Solution:

90

Exercise:

Problem: $\frac{8(6+20)}{4} + \frac{3(6+16)}{11}$

Exercise:

Problem: $\frac{1+16-3}{7} + 5(12)$

Solution:

62

Exercise:

Problem: $1^6 + 0^8 + 5^2(2 + 8)^3$

Exercise:

Problem: $\frac{5(8^2-9\cdot6)}{2^5-7} + \frac{7^2-4^2}{2^4-5}$

Solution:

5

Exercise:

Problem: $6\{2 \cdot 8 + 3\} - 5 \cdot 2 + \frac{8}{4} + (1 + 8) \cdot (1 + 11)$

Exercise:

Problem: $26 - 2 \cdot \left\{ \frac{6+20}{13} \right\}$

Solution:

22

Exercise:

Problem: $(10 + 5) \cdot (10 + 5) - 4 \cdot (60 - 4)$

Exercise:

Problem: $\frac{6^2-1}{2^3-3} + \frac{4^3+2 \cdot 3}{2 \cdot 5}$

Solution:

14

Exercise:

Problem: $\frac{(2+1)^3+2^3+1^{10}}{6^2} - \frac{15^2-[2 \cdot 5]^2}{5 \cdot 5^2}$

Ratios and Proportions
Ratios and Proportions

Ratios and Fractions

A ratio describes the proportional relationship between quantities or numbers. We represent ratio of two quantities a and b as:

"a to b" or "a : b" or $\frac{a}{b}$

Although the ratio of a and b can be expressed as a fraction $\frac{a}{b}$, the numerator and denominators here do not represent the "part to whole" relationship of that of a fraction.

For example, if we have 5 oranges and 15 apples in a basket, the ratio of apples to oranges is 5:15 which can be expressed as $\frac{5}{15}$. This ratio can be reduced to $\frac{1}{3}$ and expressed as 1:3. However, the fraction of oranges in the basket is $\frac{5}{20}$ which simplifies to $\frac{1}{4}$. The denominator of the fraction is 20 because there are a total of 20 fruit items (5 oranges and 15 apples) in the basket.

Ratios can describe more than two quantities. For example, if there are 5 oranges, 15 apples, and 6 bananas in a basket, the ratio of oranges to apples to bananas would be 5:15:6.

Proportions

Proportion describes the relationship between part to whole. Proportions can be expressed as a fraction, a decimal, relative frequency, or a percentage.

For the case of 5 oranges and 15 apples in a basket, the proportion of oranges is given by $\frac{5}{20}$ which is equivalent to $\frac{1}{4}$.

Since proportions can be represented by percents, it is often useful to asks this question: What percent is this quantity? In the above example, we could

ask, "what percent of the fruits is oranges?" To obtain the answer, we first evaluate $\frac{5}{20}$ which equals 0.25. This gives the decimal representation of the percentage of the oranges and is also called the "relative frequency." To obtain our final answer, we multiply 0.25 by 100% giving us 25%.

Proportions in Statistics

POPULATION Proportion
The proportion is the number of individuals in a population sharing a certain trait, x, divided by the total number in the population, N.

SAMPLE Proportion
A sample proportion is the proportion of individuals in a sample sharing a certain trait, divided by the total number in the sample.

Additional Resources

Rations and Proportions: An Introduction to Applied Epidemiology and Biostatistics by CDC

What’s the Difference Between a Ratio and a Fraction, and Why Should I Care?: SERP Institute

Review of Percents
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- The Meaning of Percent
- Converting A Fraction To A Percent
- Converting A Decimal To A Percent
- Converting A Percent To A Decimal

The Meaning of Percent

The word **percent** comes from the Latin word "per centum," "per" meaning "for each," and "centum" meaning "hundred."

Percent (%)
Percent means "for each hundred" or "for every hundred." The symbol % is used to represent the word percent.

Thus, $1\% = \frac{1}{100}$ or $1\% = 0.01.$

Converting A Fraction To A Percent

We can see how a fraction can be converted to a percent by analyzing the method that $\frac{3}{5}$ is converted to a percent. In order to convert $\frac{3}{5}$ to a percent, we need to introduce $\frac{1}{100}$ (since percent means for each hundred).

Example:

$$\begin{aligned} \frac{3}{5} &= \frac{3}{5} \cdot \frac{100}{100} && \text{Multiply the fraction by 1.} \\ &= \frac{3}{5} \cdot 100 \cdot \frac{1}{100} && \text{Since } \frac{100}{100} = 100 \cdot \frac{1}{100}. \\ &= \frac{300}{5} \cdot \frac{1}{100} && \text{Divide 300 by 5.} \\ &= 60 \cdot \frac{1}{100} && \text{Multiply the fractions.} \\ &= 60\% && \text{Replace } \frac{1}{100} \text{ with the } \% \text{ symbol.} \end{aligned}$$

Fraction to Percent
To convert a fraction to a percent, multiply the fraction by 1 in the form $100 \cdot \frac{1}{100}$, then replace $\frac{1}{100}$ with the % symbol.

Sample Set A

Convert each fraction to a percent.

Example:

$$\begin{aligned} \frac{1}{4} &= \frac{1}{4} \cdot 100 \cdot \frac{1}{100} \\ &= \frac{100}{4} \cdot \frac{1}{100} \\ &= 25 \cdot \frac{1}{100} \\ &= 25\% \end{aligned}$$

Example:

$$\begin{aligned} \frac{8}{5} &= \frac{8}{5} \cdot 100 \cdot \frac{1}{100} \\ &= \frac{800}{5} \cdot \frac{1}{100} \\ &= 160\% \end{aligned}$$

Example:

$$\begin{aligned} \frac{4}{9} &= \frac{4}{9} \cdot 100 \cdot \frac{1}{100} \\ &= \frac{400}{9} \cdot \frac{1}{100} \\ &= (44.4...) \cdot \frac{1}{100} \\ &= \left(44.\overline{4}\right) \cdot \frac{1}{100} \\ &= 44.\overline{4}\% \end{aligned}$$

Converting A Decimal To A Percent

We can see how a decimal is converted to a percent by analyzing the method that 0.75 is converted to a percent. We need to introduce $\frac{1}{100}$.

$$\begin{aligned} 0.75 &= 0.75 \cdot 100 \cdot \frac{1}{100} && \text{Multiply the decimal by 1.} \\ &= 75 \cdot \frac{1}{100} \\ &= 75\% && \text{Replace } \frac{1}{100} \text{ with the \% symbol.} \end{aligned}$$

Decimal to Percent
To convert a fraction to a percent, multiply the decimal by 1 in the form $100 \cdot \frac{1}{100}$, then replace $\frac{1}{100}$ with the % symbol. This amounts to moving the decimal point 2 places to the right.

Sample Set B

Convert each decimal to a percent.

Example:

$$\begin{aligned} 0.62 &= 0.62 \cdot 100 \cdot \frac{1}{100} \\ &= 62 \cdot \frac{1}{100} \\ &= 62\% \end{aligned}$$

Notice that the decimal point in the original number has been moved to the right 2 places.

Example:

$$\begin{aligned} 8.4 &= 8.4 \cdot 100 \cdot \frac{1}{100} \\ &= 840 \cdot \frac{1}{100} \\ &= 840\% \end{aligned}$$

Notice that the decimal point in the original number has been moved to the right 2 places.

Example:

$$\begin{aligned} 0.47623 &= 0.47623 \cdot 100 \cdot \frac{1}{100} \\ &= 0.47623 \cdot \frac{1}{100} \\ &= 47.623\% \end{aligned}$$

Notice that the decimal point in the original number has been moved to the right 2 places.

Converting A Percent To A Decimal

We can see how a percent is converted to a decimal by analyzing the method that 12% is converted to a decimal. We need to introduce $\frac{1}{100}$.

$$\begin{aligned} 12\% &= 12 \cdot \frac{1}{100} && \text{Replace \% with } \frac{1}{100}. \\ &= \frac{12}{100} && \text{Multiply the fractions.} \\ &= 0.12 && \text{Divide 12 by 100.} \end{aligned}$$

Percent to Decimal
To convert a percent to a decimal, replace the % symbol with $\frac{1}{100}$, then divide the number by 100. This amounts to moving the decimal point 2 places to the left.

Sample Set C

Convert each percent to a decimal.

Example:

$$\begin{aligned} 48\% &= 48 \cdot \tfrac{1}{100} \\ &= \tfrac{48}{100} \\ &= 0.48 \end{aligned}$$

Notice that the decimal point in the original number has been moved to the left 2 places.

Example:

$$\begin{aligned} 659\% &= 659 \cdot \tfrac{1}{100} \\ &= \tfrac{659}{100} \\ &= 6.59 \end{aligned}$$

Notice that the decimal point in the original number has been moved to the left 2 places.

Example:

$$\begin{aligned} 0.4113\% &= 0.4113 \cdot \tfrac{1}{100} \\ &= \tfrac{0.4113}{100} \\ &= 0.004113 \end{aligned}$$

Notice that the decimal point in the original number has been moved to the left 2 places.

Key Concepts

- **Convert a percent to a fraction.**

  Write the percent as a ratio with the denominator 100.
  Simplify the fraction if possible.

- **Convert a percent to a decimal.**

  Write the percent as a ratio with the denominator 100.
  Convert the fraction to a decimal by dividing the numerator by the denominator.

- **Convert a decimal to a percent.**

  Write the decimal as a fraction.

If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
Write this ratio as a percent.

- **Convert a fraction to a percent.**

  Convert the fraction to a decimal.
  Convert the decimal to a percent.

Exercises

For the following problems, convert each fraction to a percent.
Exercise:

Problem: $\frac{2}{5}$

Solution:

40%

Exercise:

Problem: $\frac{7}{8}$

Exercise:

Problem: $\frac{1}{8}$

Solution:

12.5%

Exercise:

Problem: $\frac{5}{16}$

Exercise:

Problem: $15 \div 22$

Solution:

68.18%

Exercise:

Problem: $\frac{2}{11}$

Exercise:

Problem: $\frac{2}{9}$

Solution:

22.22%

Exercise:

Problem: $\frac{16}{45}$

Exercise:

Problem: $\frac{27}{55}$

Solution:

49.09%

Exercise:

Problem: $\frac{7}{27}$

Exercise:

Problem: 15

Solution:

1500%

Exercise:

Problem: 8

For the following problems, convert each decimal to a percent.

Exercise:

Problem: 0.36

Solution:

36%

Exercise:

Problem: 0.42

Exercise:

Problem: 0.446

Solution:

44.6%

Exercise:

Problem: 0.1298

Exercise:

Problem: 4.25

Solution:

425%

Exercise:

Problem:5.875

Exercise:

Problem:86.98

Solution:

8698%

Exercise:

Problem:21.26

Exercise:

Problem:14

Solution:

1400%

Exercise:

Problem:12

For the following problems, convert each percent to a decimal.

Exercise:

Problem:35%

Solution:

0.35

Exercise:

Problem:76%

Exercise:

Problem:18.6%

Solution:

0.186

Exercise:

Problem:67.2%

Exercise:

Problem:9.0145%

Solution:

0.090145

Exercise:

Problem:3.00156%

Exercise:

Problem:0.00005%

Solution:

0.0000005

Exercise:

Problem:0.00034%

Convert as indicated.

Exercise:

Problem:

There are four suits of cards in a deck of cards—hearts, diamonds, clubs, and spades. The probability of randomly choosing a heart from a shuffled deck of cards is 25%. Convert the percent to:

ⓐ a fraction
ⓑ a decimal

[missing_resource: https://cnx.org/resources/70063476b47f3135d53b1ade6b1e7dd4141f54ba/CNX_BMath_Figure_06_01_002.jpg]

(credit: Riles32807, Wikimedia Commons)

Solution:

Solution

ⓐ	
	25%
Write as a ratio with denominator 100.	$\frac{25}{100}$
Simplify.	$\frac{1}{4}$

ⓑ	$\frac{1}{4}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.25

Solve Percent Applications
By the end of this section, you will be able to:

- Translate and solve basic percent equations
- Solve percent applications
- Find percent increase and percent decrease
- Solve simple interest applications
- Solve applications with discount or mark-up

Note:
Before you get started, take this readiness quiz.

1. Convert 4.5% to a decimal.
   If you missed this problem, review [link].
2. Convert 0.6 to a percent.
   If you missed this problem, review [link].
3. Round 0.875 to the nearest hundredth.
   If you missed this problem, review [link].
4. Multiply (4.5)(2.38).
   If you missed this problem, review [link].
5. Solve $3.5 = 0.7n$.
   If you missed this problem, review [link].
6. Subtract $50 - 37.45$.
   If you missed this problem, review [link].

Translate and Solve Basic Percent Equations

We will solve percent equations using the methods we used to solve equations with fractions or decimals. Without the tools of algebra, the best method available to solve percent problems was by setting them up as proportions. Now as an algebra student, you can just translate English sentences into algebraic equations and then solve the equations.

We can use any letter you like as a variable, but it is a good idea to choose a letter that will remind us of what you are looking for. We must be sure to change the given percent to a decimal when we put it in the equation.

Example:
Exercise:

Problem: Translate and solve: What number is 35% of 90?

Solution:
Solution

	$\underbrace{\text{What number}}\ \underbrace{\text{is}}\ \underbrace{35\%}\ \underbrace{\text{of}}\ \underbrace{90?}$
Translate into algebra. Let n= the number.	$n = 0.35 \cdot 90$
Remember "of" means multiply, "is" means equals.	
Multiply.	$n = 31.5$
	31.5 is 35% of 90

Note:
Exercise:

Problem: Translate and solve:

What number is 45% of 80?

Solution:

36

Note:
Exercise:

Problem: Translate and solve:

What number is 55% of 60?

Solution:

33

We must be very careful when we translate the words in the next example. The unknown quantity will not be isolated at first, like it was in [link]. We will again use direct translation to write the equation.

Example:
Exercise:

Problem: Translate and solve: 6.5% of what number is $1.17?

Solution:
Solution

	$\underbrace{6.5\%}\ \underbrace{\text{of}}\ \underbrace{\text{what number}}\ \underbrace{\text{is}}\ \underbrace{\$1.17?}$
Translate. Let $n =$ the number.	$0.065 \cdot n = 1.17$
Multiply.	$0.065n = 1.17$
Divide both sides by 0.065 and simplify.	$n = 18$
	6.5% of \$18 is \$1.17

Note:
Exercise:

Problem: Translate and solve:

7.5% of what number is $1.95?

Solution:

$26

Note:
Exercise:

Problem: Translate and solve:

8.5% of what number is $3.06?

Solution:

$36

In the next example, we are looking for the percent.

Example:
Exercise:

Problem: Translate and solve: 144 is what percent of 96?

Solution:
Solution

	144 is what percent of 96?
Translate into algebra. Let $p =$ the percent.	$144 = p \cdot 96$
Multiply.	$144 = 96p$
Divide by 96 and simplify.	$1.5 = p$
Convert to percent.	$150\% = p$
	144 is 150% of 96

Note that we are asked to find percent, so we must have our final result in percent form.

Note:

Exercise:

Problem: Translate and solve:

110 is what percent of 88?

Solution:

125%

Note:
Exercise:

Problem: Translate and solve:

126 is what percent of 72?

Solution:

175%

Solve Applications of Percent

Many applications of percent—such as tips, sales tax, discounts, and interest—occur in our daily lives. To solve these applications we'll translate to a basic percent equation, just like those we solved in previous examples. Once we translate the sentence into a percent equation, we know how to solve it.

We will restate the problem solving strategy we used earlier for easy reference.

Note:
Use a Problem-Solving Strategy to Solve an Application.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose a variable to represent that quantity.
Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Now that we have the strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications will involve everyday situations, you can rely on your own experience.

Example:
Exercise:

Problem:

Dezohn and his girlfriend enjoyed a nice dinner at a restaurant and his bill was $68.50. He wants to leave an 18% tip. If the tip will be 18% of the total bill, how much tip should he leave?

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the amount of tip should Dezohn leave
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let t = amount of tip.
Step 4. Translate into an equation.		The tip is 18% of the total bill.
Write a sentence that gives the information to find it.		$\underbrace{\text{The tip}}\ \underbrace{\text{is}}\ \underbrace{18\%}\ \underbrace{\text{of}}\ \underbrace{\$68.50}$
Translate the sentence into an equation.		$t = 0.18 \cdot 68.50$
Step 5. Solve the equation. Multiply.		$t = 12.33$
Step 6. Check. Does this make sense?		
Yes, 20% of $70 is $14.		
Step 7. Answer the question with a complete sentence.		Dezohn should leave a tip of $12.33.

Notice that we used t to represent the unknown tip.

Note:
Exercise:

Problem:

Cierra and her sister enjoyed a dinner in a restaurant and the bill was $81.50. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Solution:

$14.67

Note:
Exercise:

Problem:

Kimngoc had lunch at her favorite restaurant. She wants to leave 15% of the total bill as her tip. If her bill was $14.40, how much will she leave for the tip?

Solution:

$2.16

Example:
Exercise:

Problem:

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the total amount of potassium that is

		recommended
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let a = total amount of potassium.
Step 4. Translate. Write a sentence that gives the information to find it.		$\underbrace{\text{85 mg}}\ \underbrace{\text{is}}\ \underbrace{\text{2\%}}\ \underbrace{\text{of the}}\ \underbrace{\text{total amount}}$
Translate into an equation.		$85 = 0.02 \cdot a$
Step 5. Solve the equation.		$4{,}250 = a$
Step 6. Check. Does this make sense?		
Yes, 2% is a small percent and 85 is a small part of 4,250.		
Step 7. Answer the question with a complete sentence.		The amount of potassium that is recommended is 4,250 mg.

Note:
Exercise:

Problem:

One serving of wheat square cereal has seven grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

Solution:

25 grams

Note:
Exercise:

Problem:

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Solution:

2,375 mg

Example:
Exercise:

Problem:

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the percent of the total calories from fat
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $p =$ percent of fat.
Step 4. Translate. Write a sentence that gives the information to find it.		$\underbrace{\text{What percent}}\ \underbrace{\text{of}}\ \underbrace{480}\ \underbrace{\text{is}}\ \underbrace{240?}$
Translate into an equation.		$p \cdot 480 = 240$
Step 5. Solve the equation.		$480p = 240$
Divide by 480.		$p = 0.5$
Put in a percent form.		$p = 50\%$
Step 6. Check. Does this make sense?		
Yes, 240 is half of 480, so 50% makes sense.		
Step 7. Answer the question with a complete		Of the total calories in each

sentence.		brownie, 50% is fat.

Note:
Exercise:

Problem: Solve. Round to the nearest whole percent.

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat?

Solution:

26%

Note:
Exercise:

Problem: Solve. Round to the nearest whole percent.

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat?

Solution:

40%

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the percent increase, first we find the amount of increase, the difference of the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

Note:
Find the Percent Increase.

Find the amount of increase. $\text{new amount} - \text{original amount} = \text{increase}$
Find the percent increase. The increase is what percent of the original amount?

Example:
Exercise:

Problem:

In 2011, the California governor proposed raising community college fees from \$26 a unit to \$36 a unit. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the percent increase
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $p =$ the percent.
Step 4. Translate. Write a sentence that gives the information to find it.		
First find the amount of increase.		new amount − original amount = increase
		$36 - 26 = 10$
Find the percent.		Increase is what percent of the original amount?
		$\underbrace{10}\ \underbrace{\text{is}}\ \underbrace{\text{what percent}}\ \underbrace{\text{of}}\ \underbrace{26?}$
Translate into an equation.		$10 = p \cdot 26$
Step 5. Solve the equation.		$10 = 26p$
Divide by 26.		$0.384 = p$
Change to percent form; round to the nearest tenth.		$38.4\% = p$

Step 6. Check. Does this make sense?		
Yes, 38.4% is close to $\frac{1}{3}$, and 10 is close to $\frac{1}{3}$ of 26.		
Step 7. Answer the question with a complete sentence.		The new fees represent a 38.4% increase over the old fees.

Notice that we rounded the division to the nearest thousandth in order to round the percent to the nearest tenth.

Note:
Exercise:

Problem: Find the percent increase. (Round to the nearest tenth of a percent.)

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.

Solution:

8.8%

Note:
Exercise:

Problem: Find the percent increase.

In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was $2.25.

Solution:

50%

Finding the percent decrease is very similar to finding the percent increase, but now the amount of decrease is the difference of the original amount and the new amount. Then we find what percent the amount of decrease is of the original amount.

Note:
Find the Percent Decrease.

Find the amount of decrease. $\text{original amount} - \text{new amount} = \text{decrease}$

Find the percent decrease.The decrease is what percent of the original amount?

Example:
Exercise:

Problem:

The average price of a gallon of gas in one city in June 2014 was $3.71. The average price in that city in July was $3.64. Find the percent decrease.

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the percent decrease
Step 3. Name what we are looking for.		
Choose a variable to represent that quantity.		Let $p =$ the percent decrease.
Step 4. Translate. Write a sentence that gives the information to find it.		
First find the amount of decrease.		$3.71 - 3.64 = 0.07$
Find the percent.		Decrease is what percent of the original amaount?
		$\underbrace{0.07}\ \underbrace{\text{is}}\ \underbrace{\text{what percent}}\ \underbrace{\text{of}}\ \underbrace{3.71}?$
Translate into an equation.		$0.07 = p \cdot 3.71$
Step 5. Solve the equation.		$0.07 = 3.71p$
Divide by 3.71.		$0.019 = p$
Change to percent form; round to the nearest		

tenth.		$1.9\% = p$
Step 6. Check. Does this make sense?		
Yes, if the original price was $4, a 2% decrease would be 8 cents.		
Step 7. Answer the question with a complete sentence.		The price of gas decreased 1.9%.

Note:
Exercise:

Problem: Find the percent decrease. (Round to the nearest tenth of a percent.)

The population of North Dakota was about 672,000 in 2010. The population is projected to be about 630,000 in 2020.

Solution:

6.3%

Note:
Exercise:

Problem: Find the percent decrease.

Last year, Sheila's salary was $42,000. Because of furlough days, this year, her salary was $37,800.

Solution:

10%

Solve Simple Interest Applications

Do you know that banks pay you to keep your money? The money a customer puts in the bank is called the **principal**, P, and the money the bank pays the customer is called the **interest**. The interest is computed as a certain percent of the principal; called the **rate of interest**, r. We usually express rate of interest as a percent per year, and we calculate it by using the decimal equivalent of the percent. The variable t, (for *time*) represents the number of years the money is in the account.

To find the interest we use the simple interest formula, $I = Prt$.

Note:
Simple Interest
If an amount of money, P, called the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is
Equation:

$$I = Prt \qquad \text{where} \qquad \begin{array}{rcl} I & = & \text{interest} \\ P & = & \text{principal} \\ r & = & \text{rate} \\ t & = & \text{time} \end{array}$$

Interest earned according to this formula is called **simple interest**.

Interest may also be calculated another way, called compound interest. This type of interest will be covered in later math classes.

The formula we use to calculate simple interest is $I = Prt$. To use the formula, we substitute in the values the problem gives us for the variables, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

Example:
Exercise:

Problem:

Nathaly deposited $12,500 in her bank account where it will earn 4% interest. How much interest will Nathaly earn in 5 years?
Equation:

$$\begin{array}{rcl} I & = & ? \\ P & = & \$12{,}500 \\ r & = & 4\% \\ t & = & 5 \text{ years} \end{array}$$

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of interest earned
Step 3. Name what we are looking for. Choose a variable to represent that quantity	Let $I =$ the amount of interest.
Step 4. Translate into an equation.	
Write the formula.	$I = Prt$
Substitute in the given information.	$I = (12{,}500)(.04)(5)$
Step 5. Solve the equation.	$I = 2{,}500$
Step 6. Check: Does this make sense?	
Is \$2,500 is a reasonable interest on \$12,500? Yes.	
Step 7. Answer the question with a complete sentence.	The interest is \$2,500.

Note:
Exercise:

Problem:

Areli invested a principal of \$950 in her bank account with interest rate 3%. How much interest did she earn in 5 years?

Solution:

\$142.50

Note:
Exercise:

Problem:

Susana invested a principal of \$36,000 in her bank account with interest rate 6.5%. How much interest did she earn in 3 years?

Solution:

\$7,020

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we don't know the rate. To find the rate, we use the simple interest formula, substitute in the given values for the principal and time, and then solve for the rate.

Example:
Exercise:

Problem:

Loren loaned his brother \$3,000 to help him buy a car. In 4 years his brother paid him back the \$3,000 plus \$660 in interest. What was the rate of interest?

Equation:

$$\begin{aligned} I &= \$660 \\ P &= \$3{,}000 \\ r &= ? \\ t &= 4 \text{ years} \end{aligned}$$

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the rate of interest
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let r = rate of interest.
Step 4. Translate into an equation.	
Write the formula.	$I = Prt$
Substitute in the given information.	$660 = (3{,}000)r(4)$
Step 5. Solve the equation.	$660 = (12{,}000)r$
Divide.	$0.055 = r$
Change to percent form.	$5.5\% = r$
Step 6. Check: Does this make sense?	
Step 7. Answer the question with a complete sentence.	The rate of interest was 5.5%.

$$\begin{aligned} I &= Prt \\ 660 &\stackrel{?}{=} (3{,}000)(0.055)(4) \\ 660 &= 660\checkmark \end{aligned}$$

Notice that in this example, Loren's brother paid Loren interest, just like a bank would have paid interest if Loren invested his money there.

Note:
Exercise:

Problem:

Jim loaned his sister $5,000 to help her buy a house. In 3 years, she paid him the $5,000, plus $900 interest. What was the rate of interest?

Solution:

6%

Note:
Exercise:

Problem:

Hang borrowed $7,500 from her parents to pay her tuition. In 5 years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of interest?

Solution:

4%

Example:
Exercise:

Problem:

Eduardo noticed that his new car loan papers stated that with a 7.5% interest rate, he would pay $6,596.25 in interest over 5 years. How much did he borrow to pay for his car?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount borrowed (the principal)
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $P=$ principal borrowed.
Step 4. Translate into an equation. Write the formula. Substitute in the given information.	$$\begin{aligned} I &= Prt \\ 6{,}596.25 &= P(0.075)(5) \end{aligned}$$
Step 5. Solve the equation. Divide.	$$\begin{aligned} 6{,}596.25 &= 0.375P \\ 17{,}590 &= P \end{aligned}$$
Step 6. Check: Does this make sense? $$\begin{aligned} I &= Prt \\ 6{,}596.25 &\stackrel{?}{=} (17{,}590)(0.075)(5) \\ 6{,}596.25 &= 6{,}596.25\checkmark \end{aligned}$$	
Step 7. Answer the question with a complete sentence.	The principal was \$17,590.

Note:
Exercise:

Problem:

Sean's new car loan statement said he would pay \$4,866.25 in interest from an interest rate of 8.5% over 5 years. How much did he borrow to buy his new car?

Solution:

\$11,450

Note:
Exercise:

Problem:

In 5 years, Gloria's bank account earned \$2,400 interest at 5%. How much had she deposited in the account?

Solution:

\$9,600

Solve Applications with Discount or Mark-up

Applications of discount are very common in retail settings. When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price.

We summarize the discount model in the box below.

Note:
Discount
Equation:

$$\begin{aligned}\text{amount of discount} &= \text{discount rate} \times \text{original price}\\ \text{sale price} &= \text{original price} - \text{amount of discount}\end{aligned}$$

Keep in mind that the sale price should always be less than the original price.

Example:
Exercise:

Problem:

Elise bought a dress that was discounted 35% off of the original price of $140. What was ⓐ the amount of discount and ⓑ the sale price of the dress?

Solution:
Solution

ⓐ

$$\begin{aligned}\text{Original price} &= \$140\\ \text{Discount rate} &= 35\%\\ \text{Discount} &= ?\end{aligned}$$

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of discount
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $d =$ the amount of discount.
Step 4. Translate into an equation. Write a sentence that gives the information to find it. Translate into an equation.	The discount is 35% of $\$140$. $d = 0.35(140)$
Step 5. Solve the equation.	$d = 49$
Step 6. Check: Does this make sense? Is a \$49 discount reasonable for a \$140 dress? Yes.	
Step 7. Write a complete sentence to answer the question.	The amount of discount was \$49.

ⓑ
Read the problem again.

Step 1. Identify what we are looking for.		the sale price of the dress
Step 2. Name what we are looking for.		
Choose a variable to represent that quantity.		Let $s =$ the sale price.
Step 3. Translate into an equation.		
Write a sentence that gives the information to find it.		$\underbrace{\text{The sale price}} \; \underbrace{\text{is}} \; \underbrace{\text{the \$140}} \; \underbrace{\text{minus}} \; \underbrace{\text{the \$49 discount}}$
Translate into an equation.		$s = 140 - 49$
Step 4. Solve the equation.		$s = 91$

Step 5. Check. Does this make sense?		
Is the sale price less than the original price?		
Yes, \$91 is less than \$140.		
Step 6. Answer the question with a complete sentence.		The sale price of the dress was \$91.

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the sale price:

Sergio bought a belt that was discounted 40% from an original price of \$29.

Solution:

ⓐ \$11.60 ⓑ \$17.40

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the sale price:

Oscar bought a barbecue that was discounted 65% from an original price of \$395.

Solution:

ⓐ \$256.75 ⓑ \$138.25

There may be times when we know the original price and the sale price, and we want to know the discount rate. To find the discount rate, first we will find the amount of discount and then use it to compute the rate as a percent of the original price. [link] will show this case.

Example:
Exercise:

Problem:

Jeannette bought a swimsuit at a sale price of \$13.95. The original price of the swimsuit was \$31. Find the ⓐ amount of discount and ⓑ discount rate.

Solution:
Solution

ⓐ

$$\begin{aligned} \text{Original price} &= \$31 \\ \text{Discount} &= ? \\ \text{Sale Price} &= \$13.95 \end{aligned}$$

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of discount
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $d =$ the amount of discount.
Step 4. Translate into an equation.	
Write a sentence that gives the information to find it.	The discount is the difference between the original price and the sale price.
Translate into an equation.	$d = 31 - 13.95$
Step 5. Solve the equation.	$d = 17.05$
Step 6. Check: Does this make sense?	
Is 17.05 less than 31? Yes.	
Step 7. Answer the question with a complete sentence.	The amount of discount was \$17.05.

ⓑ
Read the problem again.

Step 1. Identify what we are looking for.		the discount rate
Step 2. Name what we are looking for.		
Choose a variable to represent it.		Let $r =$ the discount rate.
Step 3. Translate into an equation.		
Write a sentence that gives the		The discount of \$17.05 is what percent of \$31 ?

information to find it.		
Translate into an equation.		$17.05 = r \cdot 31$
Step 4. Solve the equation.		$17.05 = 31r$
Divide both sides by 31.		$0.55 = r$
Change to percent form.		$r = 55\%$
Step 5. Check. Does this make sense?		
Is $17.05 equal to 55% of $31?		
$17.05 \stackrel{?}{=} 0.55(31)$		
$17.05 = 17.05\checkmark$		
Step 6. Answer the question with a complete sentence.		The rate of discount was 55%.

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the discount rate.

Lena bought a kitchen table at the sale price of $375.20. The original price of the table was $560.

Solution:

ⓐ $184.80 ⓑ 33%

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the discount rate.

Nick bought a multi-room air conditioner at a sale price of $340. The original price of the air conditioner was $400.

Solution:

ⓐ \$60 ⓑ 15%

Applications of mark-up are very common in retail settings. The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

We summarize the mark-up model in the box below.

Note:
Mark-Up
Equation:

$$\begin{aligned}&\text{amount of mark-up} = \text{mark-up rate } \times \text{ original cost}\\&\text{list price} = \text{original cost} + \text{amount of mark up}\end{aligned}$$

Keep in mind that the list price should always be more than the original cost.

Example:
Exercise:

Problem:

Adam's art gallery bought a photograph at original cost \$250. Adam marked the price up 40%. Find the ⓐ amount of mark-up and ⓑ the list price of the photograph.

Solution:
Solution

ⓐ

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the amount of mark-up
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $m =$ the amount of markup.

Step 4. Translate into an equation.		
Write a sentence that gives the information to find it.		$\underbrace{\text{The mark-up}}\ \underbrace{\text{is}}\ \underbrace{\text{40\%}}\ \underbrace{\text{of}}\ \underbrace{\text{the \$250 original cost}}$
Translate into an equation.		$m = 0.40 \cdot 250$
Step 5. Solve the equation.		$m = 100$
Step 6. Check. Does this make sense?		
Yes, 40% is less than one-half and 100 is less than half of 250.		
Step 7. Answer the question with a complete sentence.		The mark-up on the phtograph was $100.

ⓑ

Step 1. Read the problem again.		
Step 2. Identify what we are looking for.		the list price
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $p =$ the list price.
Step 4. Translate into an equation.		
Write a sentence that gives the information to find it.		$\underbrace{\text{The list price}}\ \underbrace{\text{is}}\ \underbrace{\text{original cost}}\ \underbrace{\text{plus}}\ \underbrace{\text{the mark-up}}$
Translate into an equation.		$p = 250 + 100$
Step 5. Solve the equation.		$p = 350$
Step 6. Check. Does this make sense?		
Is the list price more than the net price?		

Is $350 more than $250? Yes.		
Step 7. Answer the question with a complete sentence.		The list price of the photograph was $350.

Note:
Exercise:

Problem: Find ⓐ the amount of mark-up and ⓑ the list price.

Jim’s music store bought a guitar at original cost $1,200. Jim marked the price up 50%.

Solution:

ⓐ $600 ⓑ $1,800

Note:
Exercise:

Problem: Find ⓐ the amount of mark-up and ⓑ the list price.

The Auto Resale Store bought Pablo’s Toyota for $8,500. They marked the price up 35%.

Solution:

ⓐ $2,975 ⓑ $11,475

Key Concepts

- **Percent Increase** To find the percent increase:

  Find the amount of increase. $\text{increase} = \text{new amount} - \text{original amount}$
  Find the percent increase. Increase is what percent of the original amount?

- **Percent Decrease** To find the percent decrease:

  Find the amount of decrease. $\text{decrease} = \text{original amount} - \text{new amount}$
  Find the percent decrease. Decrease is what percent of the original amount?

- **Simple Interest** If an amount of money, P, called the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I earned is
  Equation:

$$\begin{array}{rrcl} & I & = & Prt \\ \text{where} & I & = & \text{interest} \\ & P & = & \text{principal} \\ & r & = & \text{rate} \\ & t & = & \text{time} \end{array}$$

- **Discount**
  - amount of discount is discount rate · original price
  - sale price is original price – discount
- **Mark-up**
  - amount of mark-up is mark-up rate · original cost
  - list price is original cost + mark up

Practice Makes Perfect

Translate and Solve Basic Percent Equations

In the following exercises, translate and solve.

Exercise:

Problem: What number is 45% of 120?

Solution:

54

Exercise:

Problem: What number is 65% of 100?

Exercise:

Problem: What number is 24% of 112?

Solution:

26.88

Exercise:

Problem: What number is 36% of 124?

Exercise:

Problem: 250% of 65 is what number?

Solution:

162.5

Exercise:

Problem: 150% of 90 is what number?

Exercise:

Problem: 800% of 2250 is what number?

Solution:

18,000

Exercise:

Problem: 600% of 1740 is what number?

Exercise:

Problem: 28 is 25% of what number?

Solution:

112

Exercise:

Problem: 36 is 25% of what number?

Exercise:

Problem: 81 is 75% of what number?

Solution:

108

Exercise:

Problem: 93 is 75% of what number?

Exercise:

Problem: 8.2% of what number is $2.87?

Solution:

$35

Exercise:

Problem: 6.4% of what number is $2.88?

Exercise:

Problem: 11.5% of what number is $108.10?

Solution:

$940

Exercise:

Problem: 12.3% of what number is $92.25?

Exercise:

Problem: What percent of 260 is 78?

Solution:

30%

Exercise:

Problem: What percent of 215 is 86?

Exercise:

Problem: What percent of 1500 is 540?

Solution:

36%

Exercise:

Problem: What percent of 1800 is 846?

Exercise:

Problem: 30 is what percent of 20?

Solution:

150%

Exercise:

Problem: 50 is what percent of 40?

Exercise:

Problem: 840 is what percent of 480?

Solution:

175%

Exercise:

Problem: 790 is what percent of 395?

Solve Percent Applications

In the following exercises, solve.

Exercise:

Problem:

Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

Solution:

$11.88

Exercise:

Problem:

When Hiro and his co-workers had lunch at a restaurant near their work, the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?

Exercise:

Problem:

Trong has 12% of each paycheck automatically deposited to his savings account. His last paycheck was $2165. How much money was deposited to Trong's savings account?

Solution:

$259.80

Exercise:

Problem:

Cherise deposits 8% of each paycheck into her retirement account. Her last paycheck was $1,485. How much did Cherise deposit into her retirement account?

Exercise:

Problem:

One serving of oatmeal has eight grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

Solution:

24.2 g

Exercise:

Problem:

One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

Exercise:

Problem:

A bacon cheeseburger at a popular fast food restaurant contains 2070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

Solution:

2407 mg

Exercise:

Problem:

A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

Exercise:

Problem:

After 3 months on a diet, Lisa had lost 12% of her original weight. She lost 21 pounds. What was Lisa's original weight?

Solution:

175 lb.

Exercise:

Problem:

Tricia got a 6% raise on her weekly salary. The raise was $30 per week. What was her original salary?

Exercise:

Problem:

Yuki bought a dress on sale for $72. The sale price was 60% of the original price. What was the original price of the dress?

Solution:

$120

Exercise:

Problem:

Kim bought a pair of shoes on sale for $40.50. The sale price was 45% of the original price. What was the original price of the shoes?

Exercise:

Problem: Tim left a $9 tip for a $50 restaurant bill. What percent tip did he leave?

Solution:

18%

Exercise:

Problem: Rashid left a $15 tip for a $75 restaurant bill. What percent tip did he leave?

Exercise:

Problem:

The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

Solution:

45%

Exercise:

Problem:

The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

Exercise:

Problem:

Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?

Solution:

25%

Exercise:

Problem:

Dimple gets paid $3,200 per month. She pays $960 a month for rent. What percent of her monthly pay goes to rent?

Find Percent Increase and Percent Decrease

In the following exercises, solve.

Exercise:

Problem: Tamanika got a raise in her hourly pay, from \$15.50 to \$17.36. Find the percent increase.

Solution:

12%

Exercise:

Problem: Ayodele got a raise in her hourly pay, from \$24.50 to \$25.48. Find the percent increase.

Exercise:

Problem:

Annual student fees at the University of California rose from about \$4,000 in 2000 to about \$12,000 in 2010. Find the percent increase.

Solution:

200%

Exercise:

Problem: The price of a share of one stock rose from \$12.50 to \$50. Find the percent increase.

Exercise:

Problem:

According to *Time* magazine annual global seafood consumption rose from 22 pounds per person in the 1960s to 38 pounds per person in 2011. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:

72.7%

Exercise:

Problem:

In one month, the median home price in the Northeast rose from \$225,400 to \$241,500. Find the percent increase. (Round to the nearest tenth of a percent.)

Exercise:

Problem:

A grocery store reduced the price of a loaf of bread from \$2.80 to \$2.73. Find the percent decrease.

Solution:

2.5%

Exercise:

Problem: The price of a share of one stock fell from $8.75 to $8.54. Find the percent decrease.

Exercise:

Problem:

Hernando’s salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent decrease.

Solution:

11%

Exercise:

Problem:

In 10 years, the population of Detroit fell from 950,000 to about 712,500. Find the percent decrease.

Exercise:

Problem:

In 1 month, the median home price in the West fell from $203,400 to $192,300. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solution:

5.5%

Exercise:

Problem:

Sales of video games and consoles fell from $1,150 million to $1,030 million in 1 year. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solve Simple Interest Applications

In the following exercises, solve.

Exercise:

Problem:

Casey deposited $1,450 in a bank account with interest rate 4%. How much interest was earned in two years?

Solution:

$116

Exercise:

Problem:

Terrence deposited $5,720 in a bank account with interest rate 6%. How much interest was earned in 4 years?

Exercise:

Problem:

Robin deposited $31,000 in a bank account with interest rate 5.2%. How much interest was earned in 3 years?

Solution:

$4,836

Exercise:

Problem:

Carleen deposited $16,400 in a bank account with interest rate 3.9%. How much interest was earned in 8 years?

Exercise:

Problem:

Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of interest?

Solution:

3%

Exercise:

Problem:

Kenneth loaned his niece $1,200 to buy a computer. Two years later, she paid him back the $1,200, plus $96 interest. What was the rate of interest?

Exercise:

Problem:

Lebron loaned his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of interest?

Solution:

3.75%

Exercise:

Problem:

Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of interest?

Exercise:

Problem:

In 10 years, a bank account that paid 5.25% earned $18,375 interest. What was the principal of the account?

Solution:

$35,000

Exercise:

Problem:

In 25 years, a bond that paid 4.75% earned $2,375 interest. What was the principal of the bond?

Exercise:

Problem:

Joshua's computer loan statement said he would pay $1,244.34 in interest for a 3-year loan at 12.4%. How much did Joshua borrow to buy the computer?

Solution:

$3,345

Exercise:

Problem:

Margaret's car loan statement said she would pay $7,683.20 in interest for a 5-year loan at 9.8%. How much did Margaret borrow to buy the car?

Solve Applications with Discount or Mark-up

In the following exercises, find the sale price.

Exercise:

Problem:

Perla bought a cell phone that was on sale for $50 off. The original price of the cell phone was $189.

Solution:

$139

Exercise:

Problem: Sophie saw a dress she liked on sale for $15 off. The original price of the dress was $96.

Exercise:

Problem:

Rick wants to buy a tool set with original price $165. Next week the tool set will be on sale for $40 off.

Solution:

$125

Exercise:

Problem:

Angelo's store is having a sale on televisions. One television, with original price $859, is selling for $125 off.

In the following exercises, find ⓐ the amount of discount and ⓑ the sale price.

Exercise:

Problem: Janelle bought a beach chair on sale at 60% off. The original price was $44.95.

Solution:

ⓐ $26.97 ⓑ $17.98

Exercise:

Problem: Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.

Exercise:

Problem:

Kathy wants to buy a camera that lists for $389. The camera is on sale with a 33% discount.

Solution:

ⓐ $128.37 ⓑ $260.63

Exercise:

Problem: Colleen bought a suit that was discounted 25% from an original price of $245.

Exercise:

Problem:

Erys bought a treadmill on sale at 35% off. The original price was $949.95 (round to the nearest cent.)

Solution:

ⓐ $332.48 ⓑ $617.47

Exercise:

Problem:

Jay bought a guitar on sale at 45% off. The original price was $514.75 (round to the nearest cent.)

In the following exercises, find ⓐ the amount of discount and ⓑ the discount rate. (Round to the nearest tenth of a percent if needed.)

Exercise:

Problem:

Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.

Solution:

ⓐ $576 ⓑ 30%

Exercise:

Problem:

Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.

Exercise:

Problem:

Patty bought a baby stroller on sale for $301.75. The original price of the stroller was $355.

Solution:

ⓐ $53.25 ⓑ 15%

Exercise:

Problem: Bill found a book he wanted on sale for $20.80. The original price of the book was $32.

Exercise:

Problem:

Nikki bought a patio set on sale for $480. The original price was $850. To the nearest tenth of a percent, what was the rate of discount?

Solution:

ⓐ $370 ⓑ 43.5%

Exercise:

Problem:

Stella bought a dinette set on sale for $725. The original price was $1,299. To the nearest tenth of a percent, what was the rate of discount?

In the following exercises, find ⓐ the amount of the mark-up and ⓑ the list price.

Exercise:

Problem:

Daria bought a bracelet at original cost \$16 to sell in her handicraft store. She marked the price up 45%.

Solution:

ⓐ \$7.20 ⓑ \$23.20

Exercise:

Problem:

Regina bought a handmade quilt at original cost \$120 to sell in her quilt store. She marked the price up 55%.

Exercise:

Problem:

Tom paid \$0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up.

Solution:

ⓐ \$0.20 ⓑ \$0.80

Exercise:

Problem:

Flora paid her supplier \$0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up.

Exercise:

Problem:

Alan bought a used bicycle for \$115. After re-conditioning it, he added 225% mark-up and then advertised it for sale.

Solution:

ⓐ \$258.75 ⓑ \$373.75

Exercise:

Problem:

Michael bought a classic car for \$8,500. He restored it, then added 150% mark-up before advertising it for sale.

Everyday Math

Exercise:

Problem:

Leaving a Tip At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Solution:

21.2%

Exercise:

Problem:

Splitting a Bill Four friends went out to lunch and the bill came to $53.75. They decided to add enough tip to make a total of $64, so that they could easily split the bill evenly among themselves. What percent tip did they leave?

Writing Exercises

Exercise:

Problem:

Without solving the problem “44 is 80% of what number” think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

Solution:

The number should be greater than 44. Since 80% equals 0.8 in decimal form, 0.8 is less than one, and we must multiply the number by 0.8 to get 44, the number must be greater than 44.

Exercise:

Problem:

Without solving the problem “What is 20% of 300?” think about what the solution might be. Should it be a number that is greater than 300 or less than 300? Explain your reasoning.

Exercise:

Problem:

After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

Solution:

He meant that he should have packed half the shorts and twice the shirts.

Exercise:

Problem:

Because of road construction in one city, commuters were advised to plan that their Monday morning commute would take 150% of their usual commuting time. Explain what this means.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
translate and solve basic percent equations.			
solve percent applications.			
find percent increase and percent decrease.			
solve simple interest applications.			
solve applications with discount or mark-up.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Glossary

amount of discount
: The amount of discount is the amount resulting when a discount rate is multiplied by the original price of an item.

discount rate
: The discount rate is the percent used to determine the amount of a discount, common in retail settings.

interest
: Interest is the money that a bank pays its customers for keeping their money in the bank.

list price
: The list price is the price a retailer sells an item for.

mark-up
: A mark-up is a percentage of the original cost used to increase the price of an item.

original cost
: The original cost in a retail setting, is the price that a retailer pays for an item.

principal
: The principal is the original amount of money invested or borrowed for a period of time at a specific interest rate.

rate of interest
: The rate of interest is a percent of the principal, usually expressed as a percent per year.

simple interest

Simple interest is the interest earned according to the formula $I = Prt$.

Percentiles and Quartiles

The common measures of location are **quartiles** and **percentiles**

Quartiles are special percentiles. The first quartile, Q_1, is the same as the 25th percentile, and the third quartile, Q_3, is the same as the 75th percentile. The median, M, is called both the second quartile and the 50th percentile.

To calculate quartiles and percentiles, the data must be ordered from smallest to largest. Quartiles divide ordered data into quarters. Percentiles divide ordered data into hundredths. To score in the 90th percentile of an exam does not mean, necessarily, that you received 90% on a test. It means that 90% of test scores are the same or less than your score and 10% of the test scores are the same or greater than your test score.

Percentiles are useful for comparing values. For this reason, universities and colleges use percentiles extensively. One instance in which colleges and universities use percentiles is when SAT results are used to determine a minimum testing score that will be used as an acceptance factor. For example, suppose Duke accepts SAT scores at or above the 75th percentile. That translates into a score of at least 1220.

Percentiles are mostly used with very large populations. Therefore, if you were to say that 90% of the test scores are less (and not the same or less) than your score, it would be acceptable because removing one particular data value is not significant.

The **median** is a number that measures the "center" of the data. You can think of the median as the "middle value," but it does not actually have to be one of the observed values. It is a number that separates ordered data into halves. Half the values are the same number or smaller than the median, and half the values are the same number or larger. For example, consider the following data.
1; 11.5; 6; 7.2; 4; 8; 9; 10; 6.8; 8.3; 2; 2; 10; 1
Ordered from smallest to largest:
1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

Since there are 14 observations, the median is between the seventh value, 6.8, and the eighth value, 7.2. To find the median, add the two values together and divide by two.

Equation:

$$\frac{6.8 + 7.2}{2} = 7$$

The median is seven. Half of the values are smaller than seven and half of the values are larger than seven.

Quartiles are numbers that separate the data into quarters. Quartiles may or may not be part of the data. To find the quartiles, first find the median or second quartile. The first quartile, Q_1, is the middle value of the lower half of the data, and the third quartile, Q_3, is the middle value, or median, of the upper half of the data. To get the idea, consider the same data set:
1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

The median or **second quartile** is seven. The lower half of the data are 1, 1, 2, 2, 4, 6, 6.8. The middle value of the lower half is two.
1; 1; 2; 2; 4; 6; 6.8

The number two, which is part of the data, is the **first quartile**. One-fourth of the entire sets of values are the same as or less than two and three-fourths of the values are more than two.

The upper half of the data is 7.2, 8, 8.3, 9, 10, 10, 11.5. The middle value of the upper half is nine.

The **third quartile**, Q_3, is nine. Three-fourths (75%) of the ordered data set are less than nine. One-fourth (25%) of the ordered data set are greater than nine. The third quartile is part of the data set in this example.

The **interquartile range** is a number that indicates the spread of the middle half or the middle 50% of the data. It is the difference between the third quartile (Q_3) and the first quartile (Q_1).

$IQR = Q_3 – Q_1$

The *IQR* can help to determine potential **outliers**. **A value is suspected to be a potential outlier if it is less than (1.5)(*IQR*) below the first quartile or more than (1.5)(*IQR*) above the third quartile**. Potential outliers always require further investigation.

Note:
NOTE
A potential outlier is a data point that is significantly different from the other data points. These special data points may be errors or some kind of abnormality or they may be a key to understanding the data.

Example:
Exercise:

Problem:

For the following 13 real estate prices, calculate the *IQR* and determine if any prices are potential outliers. Prices are in dollars.
389,950; 230,500; 158,000; 479,000; 639,000; 114,950; 5,500,000; 387,000; 659,000; 529,000; 575,000; 488,800; 1,095,000

Solution:

Order the data from smallest to largest.
114,950; 158,000; 230,500; 387,000; 389,950; 479,000; 488,800; 529,000; 575,000; 639,000; 659,000; 1,095,000; 5,500,000

M = 488,800

$Q_1 = \frac{230{,}500 + 387{,}000}{2} = 308{,}750$

$Q_3 = \frac{639{,}000 + 659{,}000}{2} = 649{,}000$

IQR = 649,000 – 308,750 = 340,250

(1.5)(IQR) = (1.5)(340,250) = 510,375

Q_1 – (1.5)(IQR) = 308,750 – 510,375 = –201,625

Q_3 + (1.5)(IQR) = 649,000 + 510,375 = 1,159,375

No house price is less than –201,625. However, 5,500,000 is more than 1,159,375. Therefore, 5,500,000 is a potential **outlier**.

Note:
Try It
Exercise:

Problem:

For the following 11 salaries, calculate the IQR and determine if any salaries are outliers. The salaries are in dollars.

\$33,000 \$64,500 \$28,000 \$54,000 \$72,000 \$68,500 \$69,000 \$42,000 \$54,000 \$120,000 \$40,500

Solution:

Order the data from smallest to largest.

\$28,000 \$33,000 \$40,500 \$42,000 \$54,000 \$54,000 \$64,500 \$68,500 \$69,000 \$72,000 \$120,000

Median = \$54,000

Q_1 = \$40,500

Q_3 = \$69,000

IQR = \$69,000 – \$40,500 = \$28,500

(1.5)(IQR) = (1.5)(\$28,500) = \$42,750

Q_1 – (1.5)(IQR) = \$40,500 – \$42,750 = –\$2,250

Q_3 + (1.5)(IQR) = \$69,000 + \$42,750 = \$111,750

No salary is less than –\$2,250. However, \$120,000 is more than \$11,750, so \$120,000 is a potential outlier.

Example:
Exercise:

Problem:

For the two data sets in the test scores example, find the following:

a. The interquartile range. Compare the two interquartile ranges.
b. Any outliers in either set.

Solution:

The five number summary for the day and night classes is

	Minimum	Q_1	Median	Q_3	Maximum
Day	32	56	74.5	82.5	99
Night	25.5	78	81	89	98

a. The IQR for the day group is $Q_3 - Q_1$ = 82.5 – 56 = 26.5

The IQR for the night group is $Q_3 - Q_1 = 89 - 78 = 11$

The interquartile range (the spread or variability) for the day class is larger than the night class *IQR*. This suggests more variation will be found in the day class's class test scores.

b. Day class outliers are found using the IQR times 1.5 rule. So,

- $Q_1 - IQR(1.5) = 56 - 26.5(1.5) = 16.25$
- $Q_3 + IQR(1.5) = 82.5 + 26.5(1.5) = 122.25$

Since the minimum and maximum values for the day class are greater than 16.25 and less than 122.25, there are no outliers.

Night class outliers are calculated as:

- $Q_1 - IQR\ (1.5) = 78 - 11(1.5) = 61.5$
- Q_3 + IQR(1.5) = 89 + 11(1.5) = 105.5

For this class, any test score less than 61.5 is an outlier. Therefore, the scores of 45 and 25.5 are outliers. Since no test score is greater than 105.5, there is no upper end outlier.

Note:
Try It
Exercise:

Problem:

Find the interquartile range for the following two data sets and compare them.

Test Scores for Class *A*
69; 96; 81; 79; 65; 76; 83; 99; 89; 67; 90; 77; 85; 98; 66; 91; 77; 69; 80; 94
Test Scores for Class *B*
90; 72; 80; 92; 90; 97; 92; 75; 79; 68; 70; 80; 99; 95; 78; 73; 71; 68; 95; 100

Solution:

Class *A*

Order the data from smallest to largest.

65 66 67 69 69 76 77 77 79 80 81 83 85 89 90 91 94 96 98 99

$$\text{Median} = \frac{80+81}{2} = 80.5$$

$$Q_1 = \frac{69+76}{2} = 72.5$$

$$Q_3 = \frac{90+91}{2} = 90.5$$

IQR = 90.5 – 72.5 = 18

Class *B*

Order the data from smallest to largest.

68 68 70 71 72 73 75 78 79 80 80 90 90 92 92 95 95 97 99 100

$$\text{Median} = \frac{80+80}{2} = 80$$

$$Q_1 = \frac{72+73}{2} = 72.5$$

$$Q_3 = \frac{92+95}{2} = 93.5$$

IQR = 93.5 – 72.5 = 21

The data for Class *B* has a larger *IQR*, so the scores between Q_3 and Q_1 (middle 50%) for the data for Class *B* are more spread out and not clustered about the median.

Example:
Fifty statistics students were asked how much sleep they get per school night (rounded to the nearest hour). The results were:

AMOUNT OF SLEEP PER SCHOOL NIGHT (HOURS)	FREQUENCY	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
4	2	0.04	0.04
5	5	0.10	0.14
6	7	0.14	0.28
7	12	0.24	0.52
8	14	0.28	0.80
9	7	0.14	0.94
10	3	0.06	1.00

Find the 28th percentile. Notice the 0.28 in the "cumulative relative frequency" column. Twenty-eight percent of 50 data values is 14 values. There are 14 values less than the 28th percentile. They include the two 4s, the five 5s, and the seven 6s. The 28th percentile is between the last six and the first seven. **The 28th percentile is 6.5.**
Find the median. Look again at the "cumulative relative frequency" column and find 0.52. The median is the 50th percentile or the second quartile. 50% of 50 is 25. There are 25 values less than the median. They include the two 4s, the five 5s, the seven 6s, and eleven of the 7s. The median or 50th percentile is between the 25th, or seven, and 26th, or seven, values. **The median is seven.**
Find the third quartile. The third quartile is the same as the 75th percentile. You can "eyeball" this answer. If you look at the "cumulative relative frequency" column, you find 0.52 and 0.80. When you have all the fours, fives, sixes and sevens, you have 52% of the data. When you include all the eights, you have 80% of the data. **The 75th percentile, then, must be an eight**. Another way to look at the problem is to find 75% of 50, which is 37.5, and round up to 38. The third quartile, Q_3, is the 38th value, which is an eight. You can check this answer

by counting the values. (There are 37 values below the third quartile and 12 values above.)

Note:
Try it
Exercise:

Problem:

Forty bus drivers were asked how many hours they spend each day running their routes (rounded to the nearest hour). Find the 65^{th} percentile.

Amount of time spent on route (hours)	Frequency	Relative Frequency	Cumulative Relative Frequency
2	12	0.30	0.30
3	14	0.35	0.65
4	10	0.25	0.90
5	4	0.10	1.00

Solution:

The 65^{th} percentile is between the last three and the first four.

The 65^{th} percentile is 3.5.

Example:
Exercise:

Problem: Using [link]:

a. Find the 80^{th} percentile.
b. Find the 90^{th} percentile.
c. Find the first quartile. What is another name for the first quartile?

Solution:

Using the data from the frequency table, we have:

a. The 80^{th} percentile is between the last eight and the first nine in the table (between the 40^{th} and 41^{st} values). Therefore, we need to take the mean of the 40^{th} an 41^{st} values. The 80^{th} percentile $= \frac{8+9}{2} = 8.5$
b. The 90^{th} percentile will be the 45^{th} data value (location is 0.90(50) = 45) and the 45^{th} data value is nine.
c. Q_1 is also the 25^{th} percentile. The 25^{th} percentile location calculation: P_{25} = 0.25(50) = 12.5 ≈ 13 the 13^{th} data value. Thus, the 25th percentile is six.

Note:
Try It
Exercise:

Problem:

Refer to the [link]. Find the third quartile. What is another name for the third quartile?

Solution:

The third quartile is the 75^{th} percentile, which is four. The 65^{th} percentile is between three and four, and the 90^{th} percentile is between four and 5.75. The third quartile is between 65 and 90, so it must be four.

Note:
Collaborative Statistics
Your instructor or a member of the class will ask everyone in class how many sweaters they own. Answer the following questions:

1. How many students were surveyed?
2. What kind of sampling did you do?
3. Construct two different histograms. For each, starting value = _____ ending value = ____.
4. Find the median, first quartile, and third quartile.
5. Construct a table of the data to find the following:

   a. the 10^{th} percentile
   b. the 70^{th} percentile
   c. the percent of students who own less than four sweaters

A Formula for Finding the *k*th Percentile

If you were to do a little research, you would find several formulas for calculating the k^{th} percentile. Here is one of them.

k = the k^{th} percentile. It may or may not be part of the data.

i = the index (ranking or position of a data value)

n = the total number of data values

- Order the data from smallest to largest.
- Calculate $i = \frac{k}{100}(n+1)$
- If i is an integer, then the k^{th} percentile is the data value in the i^{th} position in the ordered set of data.
- If i is not an integer, then round i up and round i down to the nearest integers. Average the two data values in these two positions in the ordered data set. This is easier to understand in an example.

Example:

Exercise:

Problem:

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*
18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the 70^{th} percentile.
b. Find the 83^{rd} percentile.

Solution:

a.
   - k = 70
   - i = the index
   - n = 29

   $i = \frac{k}{100}(n + 1) = (\frac{70}{100})(29 + 1) = 21$. Twenty-one is an integer, and the data value in the 21^{st} position in the ordered data set is 64. The 70^{th} percentile is 64 years.

b.
   - $k = 83^{rd}$ percentile
   - i = the index
   - n = 29

   $i = \frac{k}{100}(n + 1) =)\frac{83}{100})(29 + 1) = 24.9$, which is NOT an integer. Round it down to 24 and up to 25. The age in the 24^{th} position is 71 and the age in the 25^{th} position is 72. Average 71 and 72. The 83^{rd} percentile is 71.5 years.

Note:
Try It
Exercise:

Problem:

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77
Calculate the 20th percentile and the 55th percentile.

Solution:

k = 20. Index = i = $\frac{k}{100}(n+1) = \frac{20}{100}$(29 + 1) = 6. The age in the sixth position is 27. The 20th percentile is 27 years.

k = 55. Index = i = $\frac{k}{100}(n+1) = \frac{55}{100}$(29 + 1) = 16.5. Round down to 16 and up to 17. The age in the 16th position is 52 and the age in the 17th position is 55. The average of 52 and 55 is 53.5. The 55th percentile is 53.5 years.

Note:
NOTE
You can calculate percentiles using calculators and computers. There are a variety of online calculators.

A Formula for Finding the Percentile of a Value in a Data Set

Suppose you took a test and you want to know what percentile your test score is. If you have a list of the test scores for all the students who took the test, you can use the following method to determine your percentile.

Note: This is quite different than finding the k^{th} percentile, as we saw earlier.

- Order the data from smallest to largest.
- x = the number of data values counting from the bottom of the data list up to, but not including, the data value for which you want to find the

percentile.

- y = the number of data values equal to the particular data value for which you want to find the percentile.
- n = the total number of data values.
- Calculate $k = \frac{x+0.5y}{n}(100)$. Then round k to the nearest integer.

Example:
Exercise:

Problem:

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*
18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the percentile for 58.
b. Find the percentile for 25.

Solution:

a. Counting from the bottom of the list, there are 18 data values less than 58. There is one value of 58.

   x = 18 and y = 1.
   $k = \frac{x+0.5y}{n}(100) = \frac{18+0.5(1)}{29}(100) = 63.80$, which rounds to 64.
   Therefore, 58 is the 64th percentile.
b. Counting from the bottom of the list, there are three data values less than 25. There is one value of 25.

   x = 3 and y = 1.
   $k = \frac{x+0.5y}{n}(100) = \frac{3+0.5(1)}{29}(100) = 12.07$, which rounds to 12.
   Therefore, twenty-five is the 12th percentile.

Note:
Try It
Exercise:

Problem:

Listed are 30 ages for Academy Award winning best actors in order from smallest to largest.

18; 21; 22; 25; 26; 27; 29; 30; 31, 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77
Find the percentiles for 47 and 31.

Solution:

Percentile for 47: Counting from the bottom of the list, there are 15 data values less than 47. There is one value of 47.

x = 15 and y = 1.
$k = \frac{x+0.5y}{n}(100) = \frac{15+0.5(1)}{30}(100) = 51.67$, which rounds to 52.
Therefore, 47 is the 52nd percentile.

Percentile for 31: Counting from the bottom of the list, there are eight data values less than 31. There are two values of 31.

x = 8 and y = 2.
$k = \frac{x+0.5y}{n}(100) = \frac{8+0.5(2)}{30}(100) = 30$.
Therefore, 31 is the 30th percentile.

Interpreting Percentiles, Quartiles, and Median

A percentile indicates the relative standing of a data value when data are sorted into numerical order from smallest to largest. Percentages of data values are less than or equal to the kth percentile. For example, 15% of data values are less than or equal to the 15th percentile.

- Low percentiles always correspond to lower data values.

- High percentiles always correspond to higher data values.

A percentile may or may not correspond to a value judgment about whether it is "good" or "bad." The interpretation of whether a certain percentile is "good" or "bad" depends on the context of the situation to which the data applies. In some situations, a low percentile would be considered "good;" in other contexts a high percentile might be considered "good". In many situations, there is no value judgment that applies.

Understanding how to interpret percentiles properly is important not only when describing data, but also when calculating probabilities in later chapters of this text.

Note:
Guideline
When writing the interpretation of a percentile in the context of the given data, the sentence should contain the following information.

- information about the context of the situation being considered
- the data value (value of the variable) that represents the percentile
- the percent of individuals or items with data values below the percentile
- the percent of individuals or items with data values above the percentile.

Example:
Exercise:

Problem:

On a timed math test, the first quartile for time it took to finish the exam was 35 minutes. Interpret the first quartile in the context of this situation.

Solution:

- Twenty-five percent of students finished the exam in 35 minutes or less.

- Seventy-five percent of students finished the exam in 35 minutes or more.
- A low percentile could be considered good, as finishing more quickly on a timed exam is desirable. (If you take too long, you might not be able to finish.)

Note:
Try It
Exercise:

Problem:

For the 100-meter dash, the third quartile for times for finishing the race was 11.5 seconds. Interpret the third quartile in the context of the situation.

Solution:

Twenty-five percent of runners finished the race in 11.5 seconds or more. Seventy-five percent of runners finished the race in 11.5 seconds or less. A lower percentile is good because finishing a race more quickly is desirable.

Example:
Exercise:

Problem:

On a 20 question math test, the 70^{th} percentile for number of correct answers was 16. Interpret the 70^{th} percentile in the context of this situation.

Solution:

- Seventy percent of students answered 16 or fewer questions correctly.
- Thirty percent of students answered 16 or more questions correctly.
- A higher percentile could be considered good, as answering more questions correctly is desirable.

Note:
Try It
Exercise:

Problem:

On a 60 point written assignment, the 80^{th} percentile for the number of points earned was 49. Interpret the 80^{th} percentile in the context of this situation.

Solution:

Eighty percent of students earned 49 points or fewer. Twenty percent of students earned 49 or more points. A higher percentile is good because getting more points on an assignment is desirable.

Example:
Exercise:

Problem:

At a community college, it was found that the 30^{th} percentile of credit units that students are enrolled for is seven units. Interpret the 30^{th} percentile in the context of this situation.

Solution:

- Thirty percent of students are enrolled in seven or fewer credit units.
- Seventy percent of students are enrolled in seven or more credit units.
- In this example, there is no "good" or "bad" value judgment associated with a higher or lower percentile. Students attend community college for varied reasons and needs, and their course load varies according to their needs.

Note:
Try It

Exercise:

Problem:

During a season, the 40^{th} percentile for points scored per player in a game is eight. Interpret the 40^{th} percentile in the context of this situation.

Solution:

Forty percent of players scored eight points or fewer. Sixty percent of players scored eight points or more. A higher percentile is good because getting more points in a basketball game is desirable.

Example:

Sharpe Middle School is applying for a grant that will be used to add fitness equipment to the gym. The principal surveyed 15 anonymous students to determine how many minutes a day the students spend exercising. The results from the 15 anonymous students are shown.

0 minutes; 40 minutes; 60 minutes; 30 minutes; 60 minutes
10 minutes; 45 minutes; 30 minutes; 300 minutes; 90 minutes;
30 minutes; 120 minutes; 60 minutes; 0 minutes; 20 minutes

Determine the following five values.

- Min = 0
- Q_1 = 20
- Med = 40
- Q_3 = 60
- Max = 300

If you were the principal, would you be justified in purchasing new fitness equipment? Since 75% of the students exercise for 60 minutes or less daily, and since the IQR is 40 minutes (60 – 20 = 40), we know that half of the students surveyed exercise between 20 minutes and 60 minutes daily. This seems a reasonable amount of time spent exercising, so the principal would be justified in purchasing the new equipment.

However, the principal needs to be careful. The value 300 appears to be a potential outlier.

$Q_3 + 1.5(IQR) = 60 + (1.5)(40) = 120$.

The value 300 is greater than 120 so it is a potential outlier. If we delete it and calculate the five values, we get the following values:

- Min = 0
- Q_1 = 20
- Q_3 = 60
- Max = 120

We still have 75% of the students exercising for 60 minutes or less daily and half of the students exercising between 20 and 60 minutes a day. However, 15 students is a small sample and the principal should survey more students to be sure of his survey results.

References

Cauchon, Dennis, Paul Overberg. "Census data shows minorities now a majority of U.S. births." USA Today, 2012. Available online at http://usatoday30.usatoday.com/news/nation/story/2012-05-17/minority-birthscensus/55029100/1 (accessed April 3, 2013).

Data from the United States Department of Commerce: United States Census Bureau. Available online at http://www.census.gov/ (accessed April 3, 2013).

"1990 Census." United States Department of Commerce: United States Census Bureau. Available online at http://www.census.gov/main/www/cen1990.html (accessed April 3, 2013).

Data from *San Jose Mercury News*.

Data from *Time Magazine*; survey by Yankelovich Partners, Inc.

Section Review

The values that divide a rank-ordered set of data into 100 equal parts are called percentiles. Percentiles are used to compare and interpret data. For example, an observation at the 50th percentile would be greater than 50 percent of the other obeservations in the set. Quartiles divide data into quarters. The first quartile (Q_1) is the 25th percentile,the second quartile (Q_2 or median) is 50th percentile, and the

third quartile (Q_3) is the the 75$^{\text{th}}$ percentile. The interquartile range, or *IQR*, is the range of the middle 50 percent of the data values. The *IQR* is found by subtracting Q_1 from Q_3, and can help determine outliers by using the following two expressions.

- $Q_3 + IQR(1.5)$
- $Q_1 - IQR(1.5)$

Formula Review

$i = \left(\frac{k}{100}\right)(n+1)$

where i = the ranking or position of a data value,

k = the k^{th} percentile,

n = total number of data.

Formula for finding the percentile of a data value:
$k = \left(\frac{x + 0.5y}{n}\right)(100)$

where k is always rounded to the nearest integer,

x = the number of values counting from the bottom of the data list up to but not including the particular data value for which you want to find the percentile,

y = the number of data values equal to the particular data value for which you want to find the percentile,

and n = total number of data values.
Exercise:

Problem:

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the 40^{th} percentile.
b. Find the 78^{th} percentile.

Solution:

a. The 40^{th} percentile is 37 years.
b. The 78^{th} percentile is 70 years.

Exercise:

Problem:

Listed are 32 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 18; 21; 22; 25; 26; 27; 29; 30; 31; 31; 33; 36; 37; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the percentile of 37.
b. Find the percentile of 72.

Exercise:

Problem:

Jesse was ranked 37^{th} in his graduating class of 180 students. At what percentile is Jesse's ranking?

Solution:

Jesse graduated 37^{th} out of a class of 180 students. There are 180 – 37 = 143 students ranked below Jesse. There is one rank of 37.

x = 143 and y = 1.
$k = \frac{x+0.5y}{n}(100) = \frac{143+0.5(1)}{180}(100) = 79.72$, which rounds to 80.
Jesse's rank of 37 puts him at the 80^{th} percentile.

Exercise:

Problem:

a. For runners in a race, a low time means a faster run. The winners in a race have the shortest running times. Is it more desirable to have a finish time with a high or a low percentile when running a race?
b. The 20^{th} percentile of run times in a particular race is 5.2 minutes. Write a sentence interpreting the 20^{th} percentile in the context of the situation.
c. A bicyclist in the 90^{th} percentile of a bicycle race completed the race in 1 hour and 12 minutes. Is he among the fastest or slowest cyclists in the race? Write a sentence interpreting the 90^{th} percentile in the context of the situation.

Exercise:

Problem:

a. For runners in a race, a higher speed means a faster run. Is it more desirable to have a speed with a high or a low percentile when running a race?
b. The 40^{th} percentile of speeds in a particular race is 7.5 miles per hour. Write a sentence interpreting the 40^{th} percentile in the context of the situation.

Solution:

a. For runners in a race it is more desirable to have a high percentile for speed. A high percentile means a higher speed which is faster.
b. 40% of runners ran at speeds of 7.5 miles per hour or less (slower). 60% of runners ran at speeds of 7.5 miles per hour or more (faster).

Exercise:

Problem:

On an exam, would it be more desirable to earn a grade with a high or low percentile? Explain.

Exercise:

Problem:

Mina is waiting in line at the Department of Motor Vehicles (DMV). Her wait time of 32 minutes is the 85th percentile of wait times. Is that good or bad? Write a sentence interpreting the 85th percentile in the context of this situation.

Solution:

When waiting in line at the DMV, the 85th percentile would be a long wait time compared to the other people waiting. 85% of people had shorter wait times than Mina. In this context, Mina would prefer a wait time corresponding to a lower percentile. 85% of people at the DMV waited 32 minutes or less. 15% of people at the DMV waited 32 minutes or longer.

Exercise:

Problem:

In a survey collecting data about the salaries earned by recent college graduates, Li found that her salary was in the 78th percentile. Should Li be pleased or upset by this result? Explain.

Exercise:

Problem:

In a study collecting data about the repair costs of damage to automobiles in a certain type of crash tests, a certain model of car had $1,700 in damage and was in the 90th percentile. Should the manufacturer and the consumer be pleased or upset by this result? Explain and write a sentence that interprets the 90th percentile in the context of this problem.

Solution:

The manufacturer and the consumer would be upset. This is a large repair cost for the damages, compared to the other cars in the sample. INTERPRETATION: 90% of the crash tested cars had damage repair costs of $1700 or less; only 10% had damage repair costs of $1700 or more.

Exercise:

Problem:

The University of California has two criteria used to set admission standards for freshman to be admitted to a college in the UC system:

a. Students' GPAs and scores on standardized tests (SATs and ACTs) are entered into a formula that calculates an "admissions index" score. The admissions index score is used to set eligibility standards intended to meet the goal of admitting the top 12% of high school students in the state. In this context, what percentile does the top 12% represent?
b. Students whose GPAs are at or above the 96^{th} percentile of all students at their high school are eligible (called eligible in the local context), even if they are not in the top 12% of all students in the state. What percentage of students from each high school are "eligible in the local context"?

Exercise:

Problem:

Suppose that you are buying a house. You and your realtor have determined that the most expensive house you can afford is the 34^{th} percentile. The 34^{th} percentile of housing prices is $240,000 in the town you want to move to. In this town, can you afford 34% of the houses or 66% of the houses?

Solution:

You can afford 34% of houses. 66% of the houses are too expensive for your budget. INTERPRETATION: 34% of houses cost $240,000 or less. 66% of houses cost $240,000 or more.

Use [link] to calculate the following values:

Exercise:

Problem: First quartile = _______

Exercise:

Problem: Second quartile = median = 50^{th} percentile = _______

Solution:

4

Exercise:

Problem: Third quartile = _______

Exercise:

Problem: Interquartile range (*IQR*) = _____ – _____ = _____

Solution:

6 – 4 = 2

Exercise:

Problem: 10^{th} percentile = _______

Exercise:

Problem: 70^{th} percentile = _______

Solution:

6

Homework

Exercise:

Problem:

The median age for U.S. blacks currently is 30.9 years; for U.S. whites it is 42.3 years.

a. Based upon this information, give two reasons why the black median age could be lower than the white median age.
b. Does the lower median age for blacks necessarily mean that blacks die younger than whites? Why or why not?

c. How might it be possible for blacks and whites to die at approximately the same age, but for the median age for whites to be higher?

Exercise:

Problem:

Six hundred adult Americans were asked by telephone poll, "What do you think constitutes a middle-class income?" The results are in the following table. Also, include left endpoint, but not the right endpoint.

Salary ($)	Relative Frequency
< 20,000	0.02
20,000–25,000	0.09
25,000–30,000	0.19
30,000–40,000	0.26
40,000–50,000	0.18
50,000–75,000	0.17
75,000–99,999	0.02
100,000+	0.01

a. What percentage of the survey answered "not sure"?
b. What percentage think that middle-class is from $25,000 to $50,000?
c. Construct a histogram of the data.

    i. Should all bars have the same width, based on the data? Why or why not?

ii. How should the <20,000 and the 100,000+ intervals be handled? Why?

d. Find the 40th and 80th percentiles
e. Construct a bar graph of the data

Solution:

a. 1 – (0.02+0.09+0.19+0.26+0.18+0.17+0.02+0.01) = 0.06
b. 0.19+0.26+0.18 = 0.63
c. Check student’s solution.

d. 40th percentile will fall between 30,000 and 40,000

80th percentile will fall between 50,000 and 75,000
e. Check student’s solution.

Exercise:

Problem: Given the following box plot:

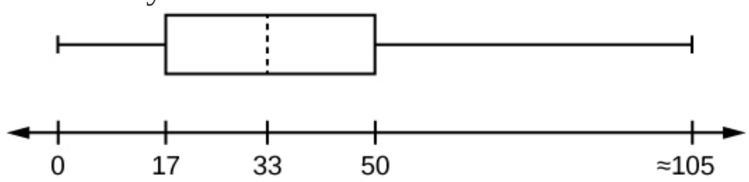
a. Which quarter has the smallest spread of data? What is that spread?
b. Which quarter has the largest spread of data? What is that spread?
c. Find the interquartile range (*IQR*).
d. Are there more data in the interval 5–10 or in the interval 10–13? How do you know this?
e. Which interval has the fewest data in it? How do you know this?

i. 0–2
ii. 2–4
iii. 10–12
iv. 12–13
v. Need more information

Exercise:

Problem:

The following box plot shows the U.S. population for 1990, the latest available year.



a. Are there fewer or more children (age 17 and under) than senior citizens (age 65 and over)? How do you know?
b. 12.6% are age 65 and over. Approximately what percentage of the population are working age adults (above age 17 to age 65)?

Solution:

a. More children; the left whisker shows that 25% of the population are children 17 and younger. The right whisker shows that 25% of the population are adults 50 and older, so adults 65 and over represent less than 25%.
b. 62.4%

Glossary

Interquartile Range
: or *IQR*, is the range of the middle 50 percent of the data values; the *IQR* is found by subtracting the first quartile from the third quartile.

Outlier
: an observation that does not fit the rest of the data

Percentile
: a number that divides ordered data into hundredths; percentiles may or may not be part of the data. The median of the data is the second quartile and the 50th percentile. The first and third quartiles are the 25th and the 75th percentiles, respectively.

Quartiles

the numbers that separate the data into quarters; quartiles may or may not be part of the data. The second quartile is the median of the data.

Review of Slope of a Line
By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{(1-4)}{(8-2)}$.
   If you missed this problem, review [link].
2. Divide: $\frac{0}{4}, \frac{4}{0}$.
   If you missed this problem, review [link].
3. Simplify: $\frac{15}{-3}, \frac{-15}{3}, \frac{-15}{-3}$.
   If you missed this problem, review [link].

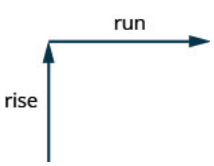
Find the Slope of a Line

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter.

In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. and as you ski or jog down a hill, you definitely experience slope.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration. Slope is a rate of change. See [link].



Note:
Slope of a Line
The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.
The rise measures the vertical change and the run measures the horizontal change.

To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

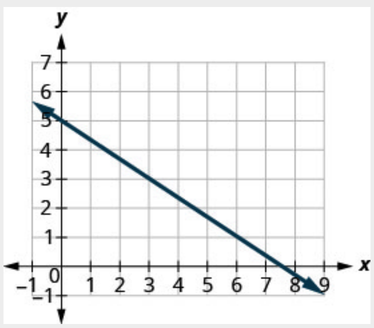
To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the *rise* and the horizontal distance is called the *run*,

Note:
Find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$.

Locate two points on the line whose coordinates are integers.
Starting with one point, sketch a right triangle, going from the first point to the second point.
Count the rise and the run on the legs of the triangle.
Take the ratio of rise to run to find the slope:$m = \frac{\text{rise}}{\text{run}}$.

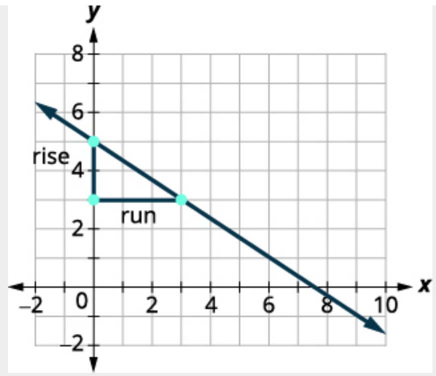
Example:
Exercise:

Problem: Find the slope of the line shown.



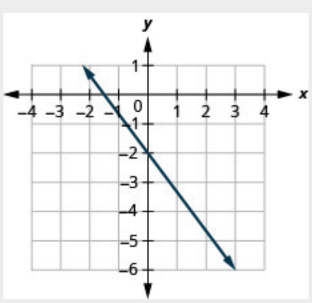
Solution:

Locate two points on the graph whose coordinates are integers.	$(0, 5)$ and $(3, 3)$
Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$ as shown in this graph.	

	
Count the rise— since it goes down, it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-2}{3}$
Simplify.	$m = -\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.
	So *y* decreases by 2 units as *x* increases by 3 units.

Note:
Exercise:

Problem: Find the slope of the line shown.

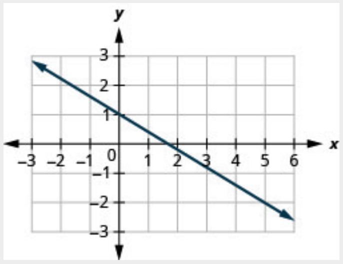


Solution:

$-\frac{4}{3}$

Note:
Exercise:

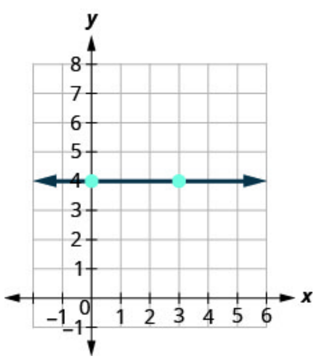
Problem: Find the slope of the line shown.



Solution:

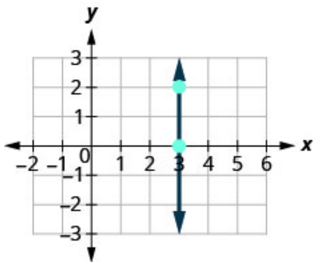
$-\frac{3}{5}$

How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4$, we could graph the line, find two points on it, and count the rise and the run. Let's see what happens when we do this, as shown in the graph below.



What is the rise?	The rise is 0.
What is the run?	The run is 3.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{0}{3}$
	$m = 0$
	The slope of the horizontal line $y = 4$ is 0.

Let's also consider a vertical line, the line $x = 3$, as shown in the graph.



What is the rise?	The rise is 2.
What is the run?	The run is 0.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{0}$

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x = 3$ is undefined.

All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

Note:
Slope of a Horizontal and Vertical Line
The slope of a horizontal line, $y = b$, is 0.
The slope of a vertical line, $x = a$, is undefined.

Example:
Exercise:

Problem: Find the slope of each line: ⓐ $x = 8$ ⓑ $y = -5$.

Solution:

ⓐ $x = 8$
This is a vertical line. Its slope is undefined.
ⓑ $y = -5$
This is a horizontal line. It has slope 0.

Note:
Exercise:

Problem: Find the slope of the line: $x = -4$.

Solution:

undefined

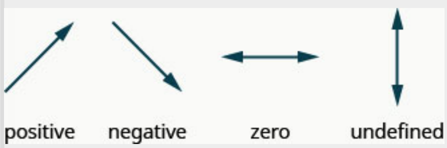
Note:
Exercise:

Problem: Find the slope of the line: $y = 7$.

Solution:

0

Note:
Quick Guide to the Slopes of Lines



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

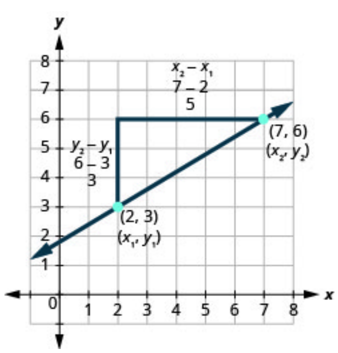
We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.
Equation:

$$\begin{array}{ll} (x_1, y_1) & \text{read "}x\text{ sub 1, }y\text{ sub 1"} \\ (x_2, y_2) & \text{read "}x\text{ sub 2, }y\text{ sub 2"} \end{array}$$

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3),(x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in this graph.



Since we have two points, we will use subscript notation. $\overset{x_1,y_1}{(2,3)}\ \overset{x_2,y_2}{(7,6)}$

$m = \frac{\text{rise}}{\text{run}}$

On the graph, we counted the rise of 3 and the run of 5. $m = \frac{3}{5}$

Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting the x-coordinates 7 and 2.

We rewrite the rise and run by putting in the coordinates. $m = \frac{6-3}{7-2}$

But 6 is y_2, the y-coordinate of the second point and 3 is y_1, the y-coordinate of the first point. So we can rewrite the slope using subscript notation. $m = \frac{y_2-y_1}{7-2}$

Also 7 is the x-coordinate of the second point and 2 is the x-coordinate of the first point. So again we rewrite the slope using subscript notation. $m = \frac{y_2-y_1}{x_2-x_1}$

We've shown that $m = \frac{y_2-y_1}{x_2-x_1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Note:
Slope of a line between two points
The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:
Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is:
Equation:

y of the second point minus y of the first point
over
x of the second point minus x of the first point.

Example:
Exercise:

Problem: Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution:

We'll call $(-2,-3)$ point #1 and $(-7,4)$ point #2.

$$\begin{pmatrix} x_1, & y_1 \\ -2, & -3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ -7, & 4 \end{pmatrix}$$

Use the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute the values.

y of the second point minus y of the first point

x of the second point minus x of the first point

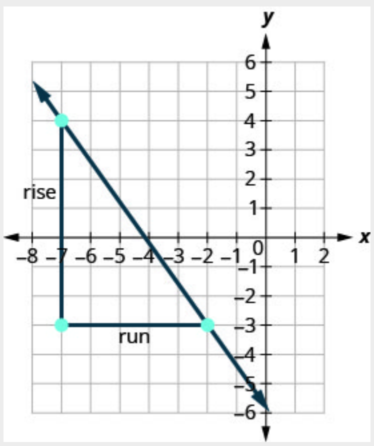
$$m = \frac{4-(-3)}{-7-(-2)}$$

$$m = \frac{7}{-5}$$

Simplify.

$$m = -\frac{7}{5}$$

Let's verify this slope on the graph shown.



Equation:

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{7}{-5}$$
$$m = -\frac{7}{5}$$

Note:
Exercise:

Problem: Use the slope formula to find the slope of the line through the pair of points: $(-3,4)$ and $(2,-1)$.

Solution:

-1

Note:
Exercise:

Problem:

Use the slope formula to find the slope of the line through the pair of points: $(-2,6)$ and $(-3,-4)$.

Solution:

10

Graph a Line Given a Point and the Slope

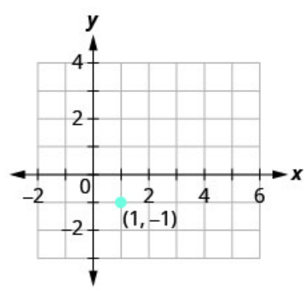
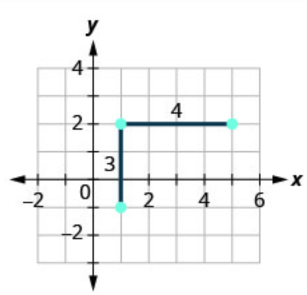
Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

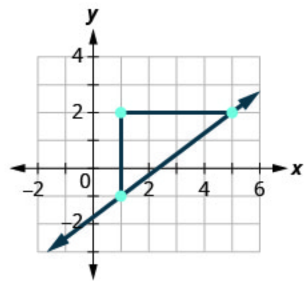
We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

Example:
How to graph a Line Given a Point and the Slope
Exercise:

Problem: Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution:

Step 1. Plot the given point.	Plot (1, –1).	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ $\text{rise} = 3$ $\text{run} = 4$
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	

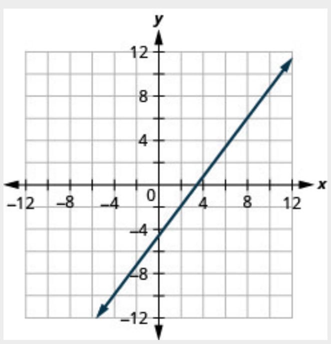
Step 4. Connect the points with a line.	Connect the two points with a line.	

You can check your work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3, and the run, -4.

Note:
Exercise:

Problem: Graph the line passing through the point $(2, -2)$ with the slope$m = \frac{4}{3}$.

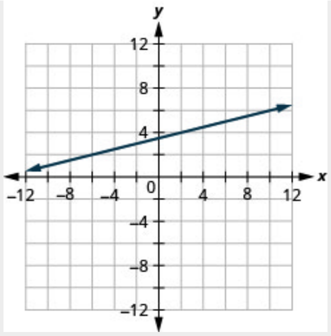
Solution:



Note:
Exercise:

Problem: Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Solution:



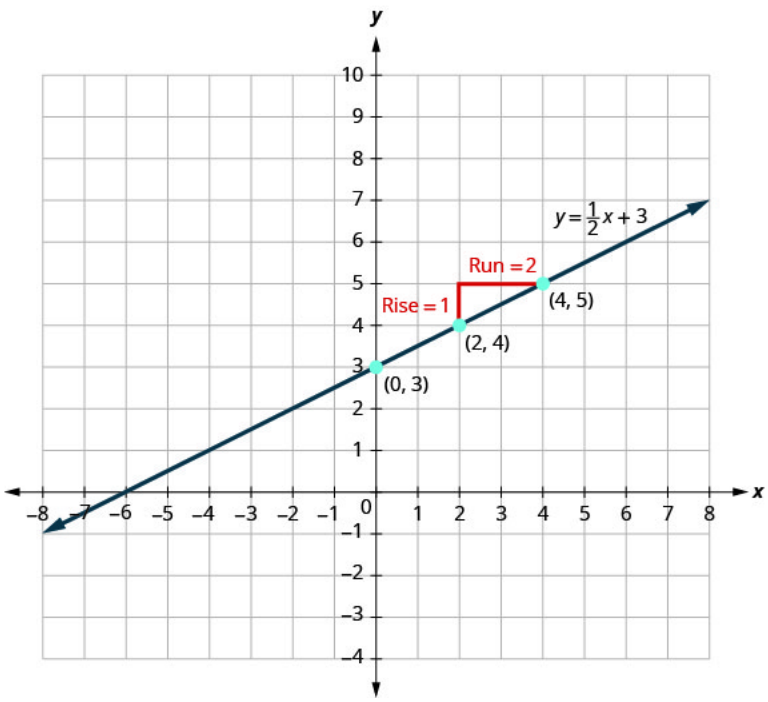
Note:
Graph a line given a point and the slope.

Plot the given point.
Use the slope formula$m = \frac{\text{rise}}{\text{run}}$to identify the rise and the run.
Starting at the given point, count out the rise and run to mark the second point.
Connect the points with a line.

Graph a Line Using its Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

See [link]. Let's look at the graph of the equation $y = \frac{1}{2}x + 3$ and find its slope and *y*-intercept.



The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:
Equation:

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{1}{2}$$

The *y*-intercept is $(0, 3)$.

Look at the equation of this line.

$y = \frac{1}{2}x + 3$

Look at the slope and *y*-intercept.

slope $m = \frac{1}{2}$ and *y*-intercept (0, 3).

When a linear equation is solved for *y*, the coefficient of the *x* term is the slope and the constant term is the *y*-coordinate of the *y*-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form. Sometimes the slope–intercept form is called the “*y*-form.”

$m = \frac{1}{2}$; *y*-intercept is (0, 3)

$y = \frac{1}{2}x + 3$

$y = mx + b$

Note:
Slope Intercept Form of an Equation of a Line
The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is $y = mx + b$.

Let's practice finding the values of the slope and *y*-intercept from the equation of a line.

Example:
Exercise:

Problem: Identify the slope and *y*-intercept of the line from the equation:

ⓐ $y = -\frac{4}{7}x - 2$ ⓑ $x + 3y = 9$

Solution:

ⓐ We compare our equation to the slope–intercept form of the equation.

Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{4}{7}x - 2$
Identify the slope.	$m = -\frac{4}{7}$
Identify the *y*-intercept.	*y*-intercept is $(0, -2)$

ⓑ When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for *y*.

Solve for *y*.	$x + 3y = 9$
Subtract *x* from each side.	$3y = -x + 9$

Divide both sides by 3.	$\frac{3y}{3} = \frac{-x+9}{3}$
Simplify.	$y = -\frac{1}{3}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{3}x + 3$
Identify the slope.	$m = -\frac{1}{3}$
Identify the *y*-intercept.	*y*-intercept is (0, 3)

Note:
Exercise:

Problem: Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y = \frac{2}{5}x - 1$ ⓑ $x + 4y = 8$

Solution:

ⓐ $m = \frac{2}{5}; (0, -1)$
ⓑ $m = -\frac{1}{4}; (0, 2)$

Note:
Exercise:

Problem: Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y = -\frac{4}{3}x + 1$ ⓑ $3x + 2y = 12$

Solution:

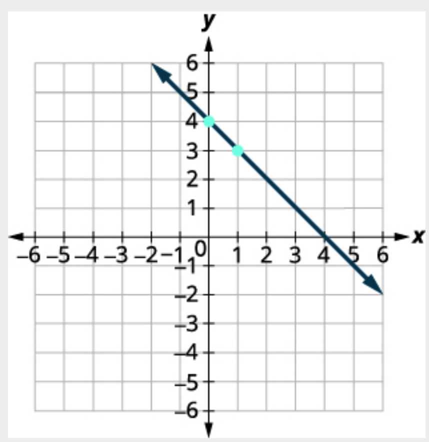
ⓐ $m = -\frac{4}{3}; (0, 1)$
ⓑ $m = -\frac{3}{2}; (0, 6)$

We have graphed a line using the slope and a point. Now that we know how to find the slope and *y*-intercept of a line from its equation, we can use the *y*-intercept as the point, and then count out the slope from there.

Example:
Exercise:

Problem: Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

Solution:

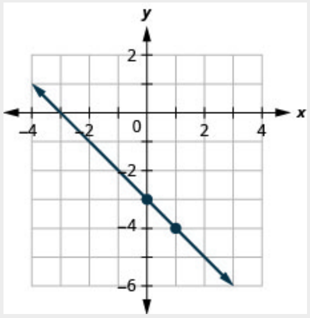
	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and *y*-intercept.	$m = -1$ *y*-intercept is $(0, 4)$
Plot the *y*-intercept.	See the graph.
Identify the rise over the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise -1, run 1 

Draw the line as shown in the graph.

Note:
Exercise:

Problem: Graph the line of the equation $y = -x - 3$ using its slope and *y*-intercept.

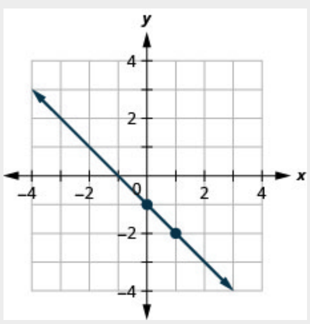
Solution:



Note:
Exercise:

Problem: Graph the line of the equation $y = -x - 1$ using its slope and *y*-intercept.

Solution:



Now that we have graphed lines by using the slope and *y*-intercept, let’s summarize all the methods we have used to graph lines.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y	$y = mx + b$	x | y; 0 | ; | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier.

Generally, plotting points is not the most efficient way to graph a line. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are five equations we graphed in this chapter, and the method we used to graph each of them.
Equation:

	Equation	**Method**
#1	$x = 2$	Vertical line
#2	$y = -1$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = -x + 4$	Slope–intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both *x* and *y* are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the *x*- intercept and $x = 0$ to find the *y*-intercept, and then found a third point by choosing another value for *x* or *y*.

Equation #5 is written in slope–intercept form. After identifying the slope and *y*-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Note:
Strategy for Choosing the Most Convenient Method to Graph a Line
Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
  - $x=a$ is a vertical line passing through the *x*-axis at *a*.
  - $y=b$ is a horizontal line passing through the *y*-axis at *b*.
- If *y* is isolated on one side of the equation, in the form $y=mx+b$, graph by using the slope and *y*-intercept.
  - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax+By=C$, find the intercepts.
  - Find the *x*- and *y*-intercepts, a third point, and then graph.

Example:
Exercise:

Problem: Determine the most convenient method to graph each line:

ⓐ $y=5$ ⓑ $4x-5y=20$ ⓒ $x=-3$ ⓓ $y=-\frac{5}{9}x+8$

Solution:

ⓐ $y=5$
This equation has only one variable, *y*. Its graph is a horizontal line crossing the *y*-axis at 5.
ⓑ $4x-5y=20$
This equation is of the form $Ax+By=C$. The easiest way to graph it will be to find the intercepts and one more point.
ⓒ $x=-3$
There is only one variable, *x*. The graph is a vertical line crossing the *x*-axis at -3.
ⓓ $y=-\frac{5}{9}x+8$
Since this equation is in $y=mx+b$ form, it will be easiest to graph this line by using the slope and *y*-intercepts.

Note:
Exercise:

Problem: Determine the most convenient method to graph each line:

ⓐ $3x+2y=12$ ⓑ $y=4$ ⓒ $y=\frac{1}{5}x-4$ ⓓ $x=-7$.

Solution:

ⓐ intercepts ⓑ horizontal line ⓒ slope-intercept ⓓ vertical line

Note:
Exercise:

Problem: Determine the most convenient method to graph each line:

ⓐ $x=6$ ⓑ $y=-\frac{3}{4}x+1$ ⓒ $y=-8$ ⓓ $4x-3y=-1$.

Solution:

ⓐ vertical line ⓑ slope-intercept ⓒ horizontal line
ⓓ intercepts

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real world situations.

Usually, when a linear equation models uses real-world data, different letters are used for the variables, instead of using only *x* and *y*. The variable names remind us of what quantities are being measured.

Also, we often will need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

Example:
Exercise:

Problem:

The equation $F=\frac{9}{5}C+32$ is used to convert temperatures, *C*, on the Celsius scale to temperatures, *F*, on the Fahrenheit scale.

ⓐ Find the Fahrenheit temperature for a Celsius temperature of 0.

ⓑ Find the Fahrenheit temperature for a Celsius temperature of 20.

ⓒ Interpret the slope and *F*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ

Find the Fahrenheit temperature for a Celsius temperature of 0.	$F=\frac{9}{5}C+32$
Find F when $C=0$.	$F=\frac{9}{5}(0)+32$
Simplify.	$F=32$

ⓑ

Find the Fahrenheit temperature for a Celsius temperature of 20.	$F=\frac{9}{5}C+32$
Find F when $C=20$.	$F=\frac{9}{5}(20)+32$
Simplify.	$F=36+32$
Simplify.	$F=68$

ⓒ

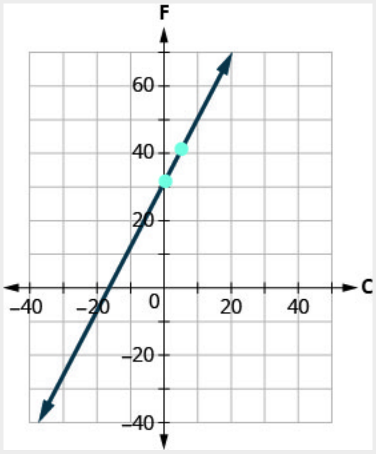
Interpret the slope and *F*-intercept of the equation.
Even though this equation uses *F* and *C*, it is still in slope–intercept form.

$y = mx + b$

$F = mC + b$

$F = \frac{9}{5}C + 32$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees.
The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.
ⓓ Graph the equation.
We'll need to use a larger scale than our usual. Start at the F-intercept $(0, 32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown in the graph.



Note:
Exercise:

Problem:

The equation $h = 2s + 50$ is used to estimate a woman's height in inches, h, based on her shoe size, s.

ⓐ Estimate the height of a child who wears women's shoe size 0.

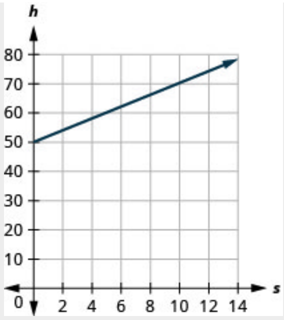
ⓑ Estimate the height of a woman with shoe size 8.

ⓒ Interpret the slope and h-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ 50 inches
ⓑ 66 inches
ⓒ The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.
ⓓ



Note:
Exercise:

Problem:

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, *T*, based on the number of cricket chirps, *n*, in one minute.

ⓐ Estimate the temperature when there are no chirps.

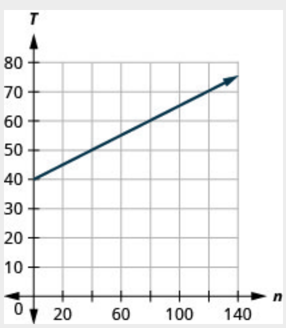
ⓑ Estimate the temperature when the number of chirps in one minute is 100.

ⓒ Interpret the slope and *T*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ 40 degrees
ⓑ 65 degrees
ⓒ The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (*F*) increases 1 degree when the number of chirps, *n*, increases by 4. The *T*-intercept means that when the number of chirps is 0, the temperature is 40°.
ⓓ



The cost of running some types business have two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

Example:
Exercise:

Problem:

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, *C*, in dollars and the number of miles, *m*, that he drives.

ⓐ Find Sam’s cost for a week when he drives 0 miles.

ⓑ Find the cost for a week when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ

Find Sam’s cost for a week when he drives 0 miles.	$C = 0.5m + 60$
Find C when $m = 0$.	$C = 0.5\,(0) + 60$
Simplify.	$C = 60$
	Sam’s costs are $60 when he drives 0 miles.

ⓑ

Find the cost for a week when he drives 250 miles.	$C = 0.5m + 60$
Find C when $m = 250$.	$C = 0.5\,(250) + 60$
Simplify.	$C = 185$
	Sam’s costs are $185 when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

$y = mx + b$

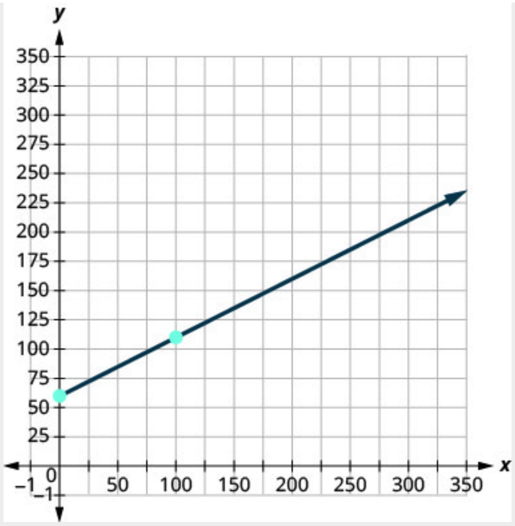
$C = 0.5p + 60$

The slope, 0.5, means that the weekly cost, *C*, increases by $0.50 when the number of miles driven, *n*, increases by 1.
The *C*-intercept means that when the number of miles driven is 0, the weekly cost is $60.
ⓓ Graph the equation.
We’ll need to use a larger scale than our usual. Start at the *C*-intercept $(0, 60)$.

To count out the slope *m* = 0.5, we rewrite it as an equivalent fraction that will make our graphing easier.

	$m = 0.5$
Rewrite as a fraction.	$m = \frac{0.5}{1}$
Multiply numerator and denominator by 100.	$m = \frac{0.5(100)}{1(100)}$
Simplify.	$m = \frac{50}{100}$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be (100, 110).



Note:
Exercise:

Problem:

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, *C*, in dollars and the number of pizzas, *p*, that she sells.

ⓐ Find Stella's cost for a week when she sells no pizzas.

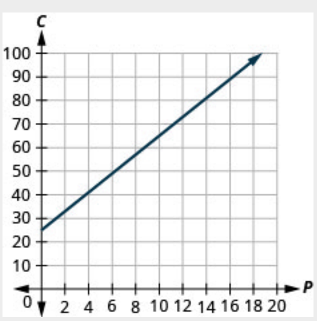
ⓑ Find the cost for a week when she sells 15 pizzas.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ \$25
ⓑ \$85
ⓒ The slope, 4, means that the weekly cost, *C*, increases by \$4 when the number of pizzas sold, *p*, increases by 1. The *C*-intercept means that when the number of pizzas sold is 0, the weekly cost is \$25.
ⓓ



Note:
Exercise:

Problem:

Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, *C*, in dollars and the number of wedding invitations, *n*, that she writes.

ⓐ Find Loreen's cost for a week when she writes no invitations.

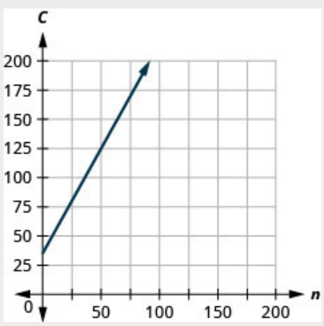
ⓑ Find the cost for a week when she writes 75 invitations.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ $35
ⓑ $170
ⓒ The slope, 1.8, means that the weekly cost, *C*, increases by $1.80 when the number of invitations, *n*, increases by 1.
The *C*-intercept means that when the number of invitations is 0, the weekly cost is $35.
ⓓ



Key Concepts

- **Slope of a Line**
  - The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.
  - The rise measures the vertical change and the run measures the horizontal change.

- **How to find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$.**

  Locate two points on the line whose coordinates are integers.
  Starting with one point, sketch a right triangle, going from the first point to the second point.
  Count the rise and the run on the legs of the triangle.
  Take the ratio of rise to run to find the slope:$m = \frac{\text{rise}}{\text{run}}$.

- **Slope of a line between two points.**
    - The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:
      Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

- **How to graph a line given a point and the slope.**

  Plot the given point.
  Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  Starting at the given point, count out the rise and run to mark the second point.
  Connect the points with a line.

- **Slope Intercept Form of an Equation of a Line**
    - The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, $(0, b)$ is $y = mx + b$

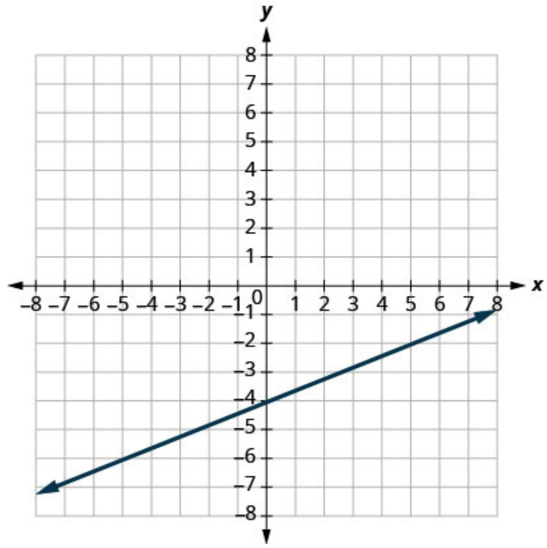
Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | *y*	$y = mx + b$	*x* | *y* 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Practice Makes Perfect

Find the Slope of a Line

In the following exercises, find the slope of each line shown.
Exercise:

Problem:

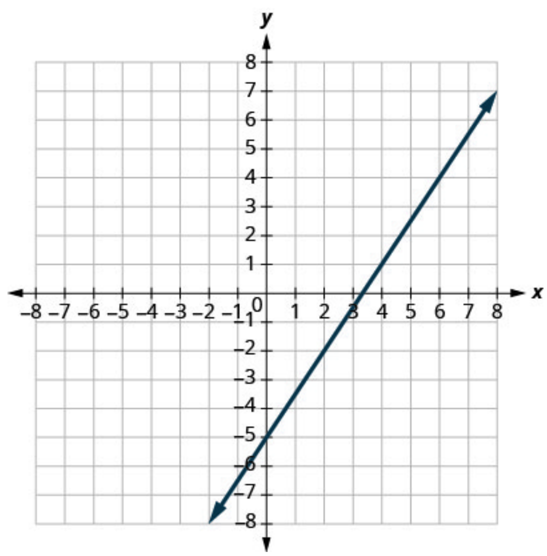


Solution:

$\frac{2}{5}$

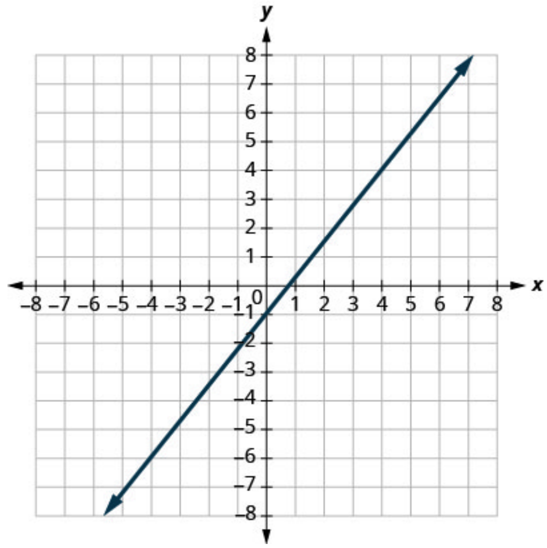
Exercise:

Problem:



Exercise:

Problem:

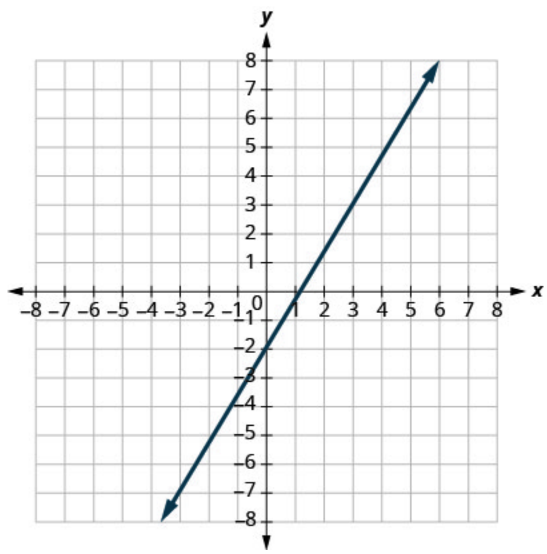


Solution:

$\frac{5}{4}$

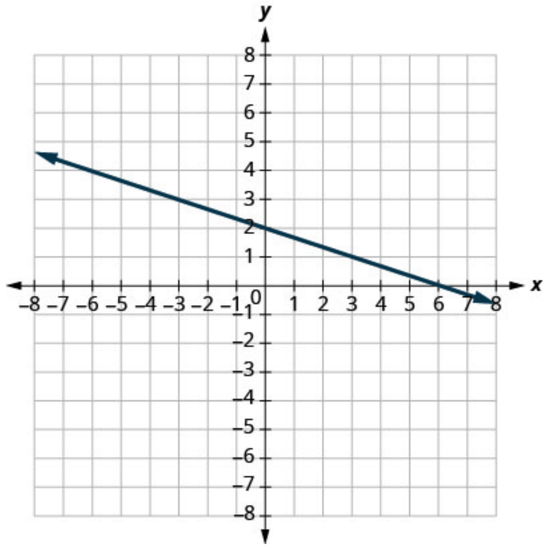
Exercise:

Problem:



Exercise:

Problem:

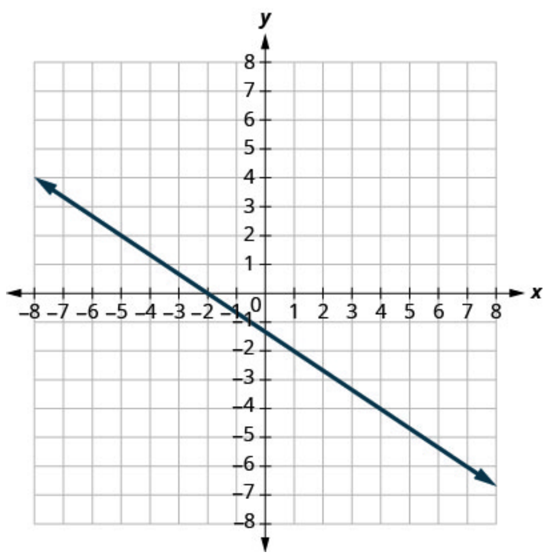


Solution:

$-\frac{1}{3}$

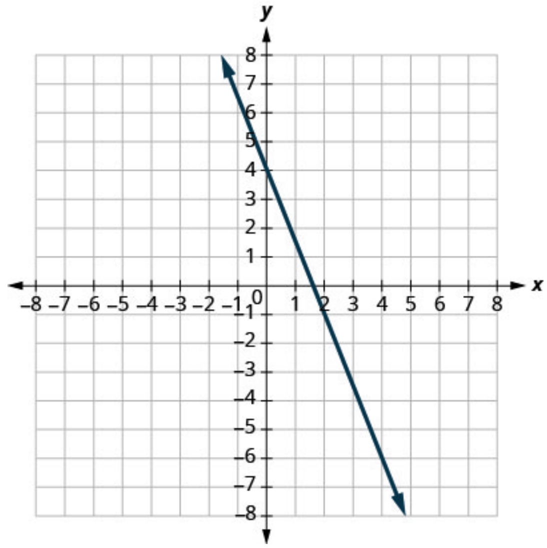
Exercise:

Problem:



Exercise:

Problem:

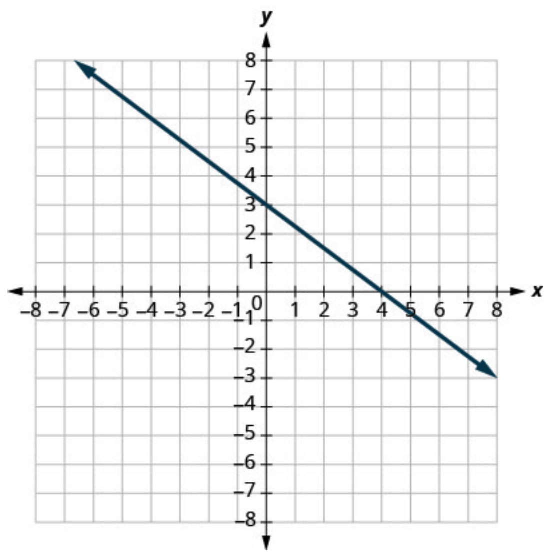


Solution:

$-\frac{5}{2}$

Exercise:

Problem:



In the following exercises, find the slope of each line.

Exercise:

Problem: $y = 3$

Solution:

0

Exercise:

Problem: $y=-2$

Exercise:

Problem: $x=-5$

Solution:

undefined

Exercise:

Problem: $x=4$

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise:

Problem: $(2,5),(4,0)$

Solution:

$-\frac{5}{2}$

Exercise:

Problem: $(3,6),(8,0)$

Exercise:

Problem: $(-3,3),(4,-5)$

Solution:

$-\frac{8}{5}$

Exercise:

Problem: $(-2,4),(3,-1)$

Exercise:

Problem: $(-1,-2),(2,5)$

Solution:

$\frac{7}{3}$

Exercise:

Problem: $(-2,-1),(6,5)$

Exercise:

Problem: $(4,-5),(1,-2)$

Solution:

-1

Exercise:

Problem: $(3,-6),(2,-2)$

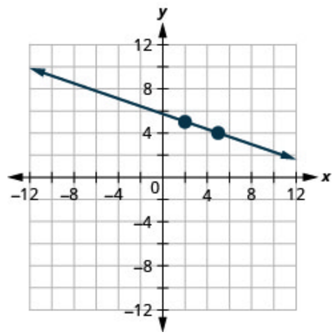
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

Exercise:

Problem: $(2,5);m=-\frac{1}{3}$

Solution:



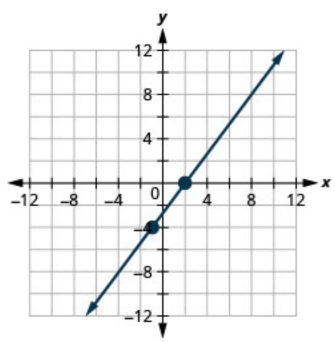
Exercise:

Problem: $(1,4);\ m=-\frac{1}{2}$

Exercise:

Problem: $(-1,-4);\ m=\frac{4}{3}$

Solution:



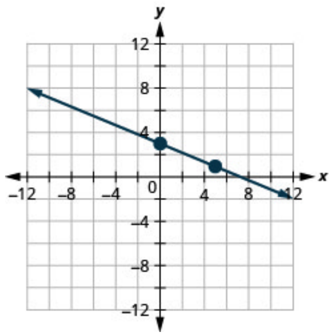
Exercise:

Problem: $(-3,-5);\ m=\frac{3}{2}$

Exercise:

Problem: y-intercept 3; $m=-\frac{2}{5}$

Solution:



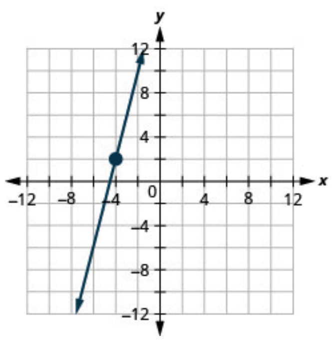
Exercise:

Problem: x-intercept -2; $m=\frac{3}{4}$

Exercise:

Problem: $(-4,2)$; $m=4$

Solution:



Exercise:

Problem: $(1,5)$; $m=-3$

Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and y-intercept of each line.

Exercise:

Problem: $y=-7x+3$

Solution:

$m=-7; (0,3)$

Exercise:

Problem: $y = 4x - 10$

Exercise:

Problem: $3x + y = 5$

Solution:

$m = -3; (0, 5)$

Exercise:

Problem: $4x + y = 8$

Exercise:

Problem: $6x + 4y = 12$

Solution:

$m = -\frac{3}{2}; (0, 3)$

Exercise:

Problem: $8x + 3y = 12$

Exercise:

Problem: $5x - 2y = 6$

Solution:

$m = \frac{5}{2}; (0, -3)$

Exercise:

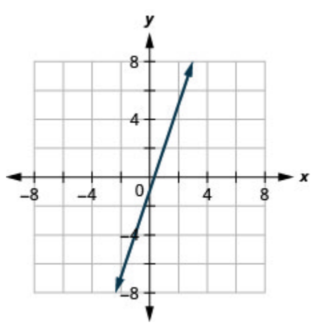
Problem: $7x - 3y = 9$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

Exercise:

Problem: $y = 3x - 1$

Solution:



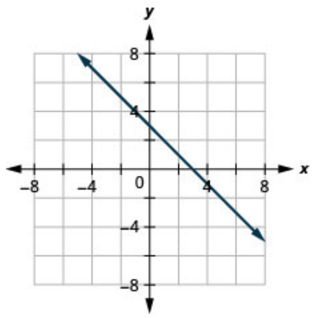
Exercise:

Problem: $y = 2x - 3$

Exercise:

Problem: $y = -x + 3$

Solution:



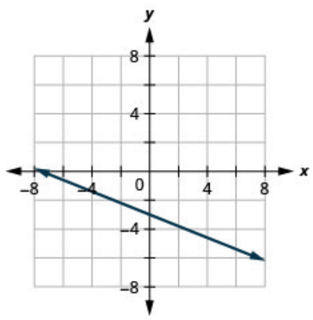
Exercise:

Problem: $y = -x - 4$

Exercise:

Problem: $y = -\frac{2}{5}x - 3$

Solution:



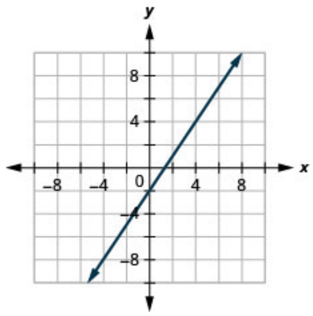
Exercise:

Problem: $y = -\frac{3}{5}x + 2$

Exercise:

Problem: $3x - 2y = 4$

Solution:



Exercise:

Problem: $3x - 4y = 8$

Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

Exercise:

Problem: $x = 2$

Solution:

vertical line

Exercise:

Problem: $y = 5$

Exercise:

Problem: $y = -3x + 4$

Solution:

slope-intercept

Exercise:

Problem: $x - y = 5$

Exercise:

Problem: $x - y = 1$

Solution:

intercepts

Exercise:

Problem: $y = \frac{2}{3}x - 1$

Exercise:

Problem: $3x - 2y = -12$

Solution:

intercepts

Exercise:

Problem: $2x - 5y = -10$

Graph and Interpret Applications of Slope–Intercept

Exercise:

Problem:

The equation $P = 31 + 1.75w$ models the relation between the amount of Tuyet's monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

ⓐ Find Tuyet's payment for a month when 0 units of water are used.

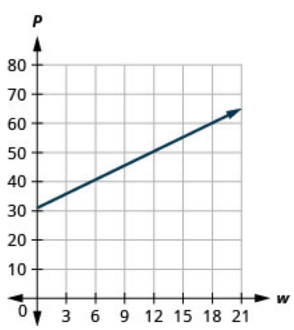
ⓑ Find Tuyet's payment for a month when 12 units of water are used.

ⓒ Interpret the slope and *P*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ $31
ⓑ $52
ⓒ The slope, 1.75, means that the payment, *P*, increases by $\$1.75$ when the number of units of water used, *w*, increases by 1. The *P*-intercept means that when the number units of water Tuyet used is 0, the payment is $31.
ⓓ



Exercise:

Problem:

The equation $P = 28 + 2.54w$ models the relation between the amount of R and y's monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

ⓐ Find the payment for a month when R and y used 0 units of water.

ⓑ Find the payment for a month when R and y used 15 units of water.

ⓒ Interpret the slope and P-intercept of the equation.

ⓓ Graph the equation.

Exercise:

Problem:

Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, R, that he is reimbursed and the number of miles, m, he drives in one day.

ⓐ Find the amount Bruce is reimbursed on a day when he drives 0 miles.

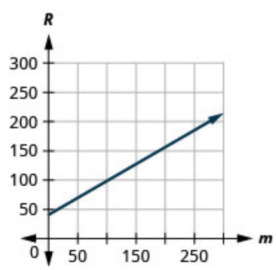
ⓑ Find the amount Bruce is reimbursed on a day when he drives 220 miles.

ⓒ Interpret the slope and R-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ \$42
ⓑ \$168.50
ⓒ The slope, 0.575 means that the amount he is reimbursed, R, increases by \$0.575 when the number of miles driven, m, increases by 1. The R-intercept means that when the number miles driven is 0, the amount reimbursed is \$42.
ⓓ



Exercise:

Problem:

Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, C, per day and the number of miles, m, she drives in one day.

ⓐ Find the cost if Janelle drives the car 0 miles one day.

ⓑ Find the cost on a day when Janelle drives the car 400 miles.

ⓒ Interpret the slope and C-intercept of the equation.

ⓓ Graph the equation.

Exercise:

Problem:

Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, S, in dollars and the amount of her sales, c, in dollars.

ⓐ Find Cherie's salary for a week when her sales were \$0.

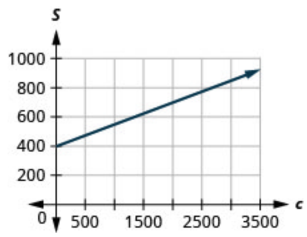
ⓑ Find Cherie's salary for a week when her sales were \$3,600.

ⓒ Interpret the slope and *S*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ \$400
ⓑ \$940
ⓒ The slope, 0.15, means that Cherie's salary, *S*, increases by \$0.15 for every \$1 increase in her sales. The *S*-intercept means that when her sales are \$0, her salary is \$400.
ⓓ



Exercise:

Problem:

Patel's weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, *S*, in dollars and the amount of his sales, *c*, in dollars.

ⓐ Find Patel's salary for a week when his sales were 0.

ⓑ Find Patel's salary for a week when his sales were 18,540.

ⓒ Interpret the slope and *S*-intercept of the equation.

ⓓ Graph the equation.

Exercise:

Problem:

Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, *C*, of the banquet and the number of guests, *g*.

ⓐ Find the cost if the number of guests is 40.

ⓑ Find the cost if the number of guests is 80.

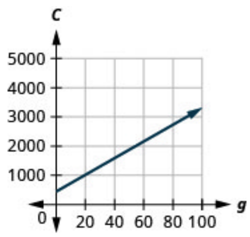
ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ \$1570
ⓑ \$5690
ⓒ The slope gives the cost per guest. The slope, 28, means that the cost, *C*, increases by \$28 when the

number of guests increases by 1. The C-intercept means that if the number of guests was 0, the cost would be \$450.
ⓓ



Exercise:

Problem:

Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, C of the banquet and the number of guests, g.

ⓐ Find the cost if the number of guests is 50.

ⓑ Find the cost if the number of guests is 100.

ⓒ Interpret the slope and C-intercept of the equation.

ⓓ Graph the equation.

Writing Exercises

Exercise:

Problem: How does the graph of a line with slope $m = \frac{1}{2}$ differ from the graph of a line with slope $m = 2$?

Solution:

Answers will vary.

Exercise:

Problem: Why is the slope of a vertical line “undefined”?

Exercise:

Problem: Explain how you can graph a line given a point and its slope.

Solution:

Answers will vary.

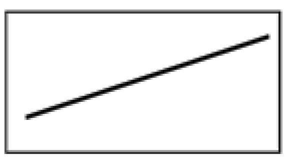
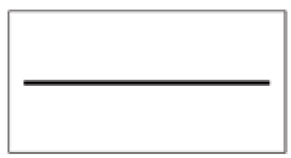
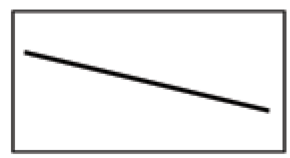
Exercise:

Problem: Explain in your own words how to decide which method to use to graph a line.

Slope and Y-Intercept of a Linear Equation
This module provides an overview of Linear Regression and Correlation: Slope and Y-Intercept of a Linear Equation as a part of Collaborative Statistics collection (col10522) by Barbara Illowsky and Susan Dean.

For the linear equation $y = a + \text{bx}$, b = slope and a = y-intercept. Note this is similar to $y = \text{mx} + b$, but it is not the same.

From algebra recall that the slope is a number that describes the steepness of a line and the y-intercept is the y coordinate of the point $(0, a)$ where the line crosses the y-axis.

If $b > 0$, the line slopes upward to the right.	If $b = 0$, the line is horizontal.	If $b < 0$, the line slopes downward to the right.
		

Three possible graphs of $y = a + \text{bx}$.

The slope is how much the response variable will increase/decrease by as the explanatory variable increases by 1. The y-intercept is the value of the response variable when the explanatory variable is 0.

Example:
A
Svetlana tutors to make extra money for college. For each tutoring session, she charges a one time fee of \$25 plus \$15 per hour of tutoring. A linear equation that expresses the total amount of money Svetlana earns for each session she tutors is $y = 25 + 15\text{x}$.

Exercise:
A

Problem:

What are the independent and dependent variables? What is the y-intercept and what is the slope? Interpret them using complete sentences.

Solution:

The independent variable (x) is the number of hours Svetlana tutors each session. The dependent variable (y) is the amount, in dollars, Svetlana earns for each session.

The y-intercept is 25 (a = 25). At the start of the tutoring session, Svetlana charges a one-time fee of $25 (this is when x = 0). The slope is 15 (b = 15). For each session, Svetlana earns $15 for each hour she tutors.

Example:
B
The equation to predict your college GPA given your high school GPA is $y = 1.11 + .66$x ,where x is your high school GPA.
Exercise:
B

Problem:

What are the explanatory and response variables? Interpret the slope and y-intercept using complete sentences.

Solution:

The explanatory variable is high school GPA. The response variable is college GPA. The slope is .66; for each additional point on your high school GPA your college GPA should increase by .66 points. The y-

intercept is 1.11; a person with a high school GPA of 0 is expected to have a college GPA of 1.1

Graphs of Discrete and Continuous Distributions
(Blank Abstract)

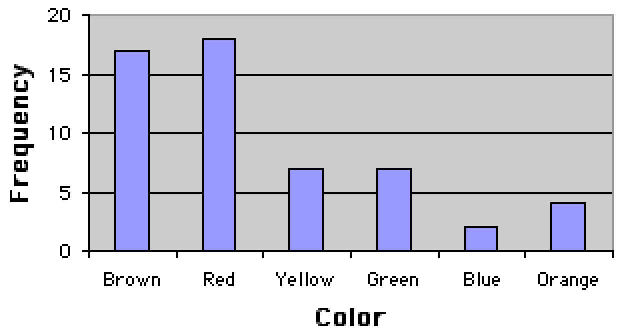
Distributions of Discrete Variables

I recently purchased a bag of Plain M&Ms. The M&M's were in six different colors. A quick count showed that there were 55 M&M's: 17 brown, 18 red, 7 yellow, 7 green, 2 blue, and 4 orange. These counts are shown below in [link].

Color	Frequency
Brown	17
Red	18
Yellow	7
Green	7
Blue	2
Orange	4

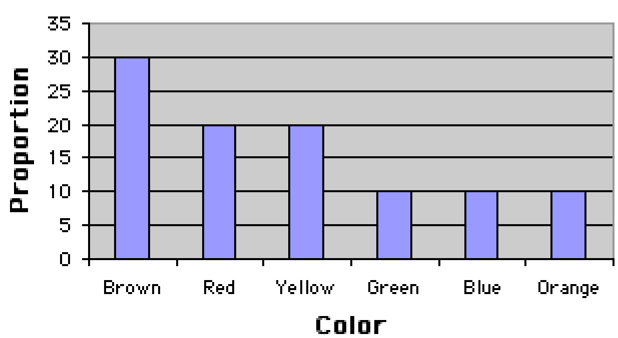
Distributions of Colors

This table is called a **frequency table** and it describes the distribution of M&M color frequencies. Not surprisingly, this kind of table is called a **frequency distribution**. Often a frequency distribution is shown graphically as in [link].



Distribution of 55 M&Ms

The distribution shown in [link] concerns just my one bag of M&M's. You might be wondering about the distribution of colors for all M&M's. The manufacturer of M&M's provides some information about this matter, but they do not tell us exactly how many M&M's of each color they have ever produced. Instead, they report proportions rather than frequencies. [link] shows these proportions. Since every M&M is one of the six familiar colors, the six proportions shown in the figure add to one. We call [link] a **probability distribution** because if you chose an M&M at random, the probability of getting, say, a brown M&M is equal to the proportion of M&M's that are brown (0.30).



Distribution of all M&Ms

Notice that the distributions in [link] and [link] are not identical. [link] portrays the distribution in a sample of 55 M&M's. [link] shows the proportions for all M&M's. Chance factors involving the machines used by the manufacturer introduce random variation into the different bags produced. Some bags will have a distribution of colors that is close to [link]; others will be farther away.

Continuous Variables

The variable "color of M&M" used in this example is a **discrete variable**, and its distributions is also called **discrete**. Let us now extend the concept of a distribution to **continuous variables**.

The data shown in [link] are the times it took one of us (DL) to move the mouse over a small target in a series of 20 trials. The times are sorted from fastest to slowest. The variable "time to respond" is a continuous variable. With time measured accurately (to many decimal places), no two response times would be expected to be the same. Measuring time in milliseconds (thousandths of a second) is often precise enough to approximate a continuous variable in Psychology. As you can see in [link], measuring DL's responses this way produced times no two of which were the same. As a result, a frequency distribution would be uninformative: it would consist of the 20 times in the experiment, each with a frequency of 1.

The solution to this problem is to create a **grouped frequency distribution**. In a grouped frequency distribution, scores falling withing various ranges are tabulated. [link] shows a grouped frequency distribution for these 20 times.

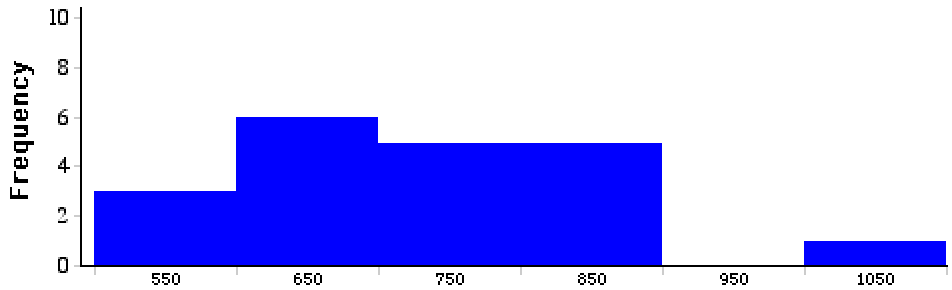
568	720
577	728
581	729
640	777
641	808
645	824
657	825
673	865
696	875
703	1007

Response Times (in milliseconds)

Range	**Frequency**
500-600	3
600-700	6
700-800	5
800-900	5
900-1000	0
1000-1100	1

Grouped frequency distribution

Grouped frequency distributions may be portrayed graphically. [link] shows a graphical representation of the frequency distribution in [link]. This kind of graph is called a **histogram**. Chapter 2 contains an entire section devoted to histograms.

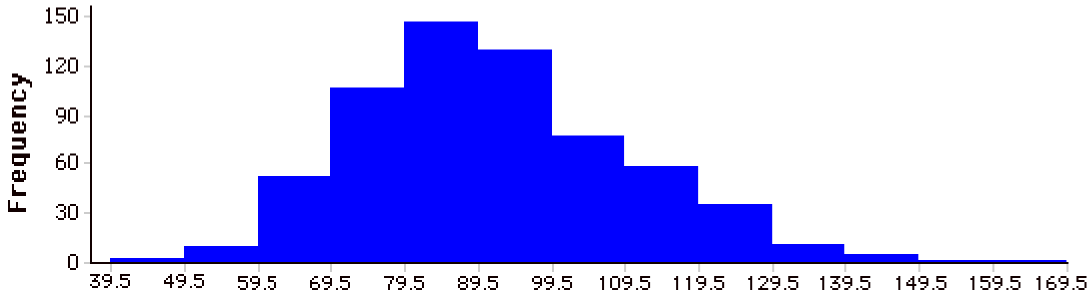


A histogram of the grouped frequency distribution shown in [link]. The labels on the X-axis are the middle values of the range they represent.

Shapes of Distributions

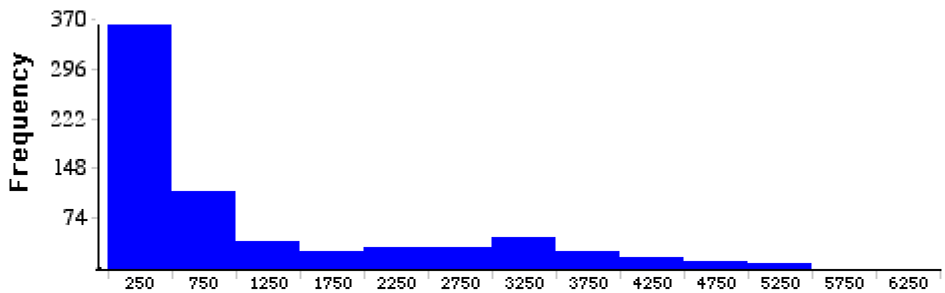
Distributions have different shapes; they don't all look like the normal distribution in [link]. For example, the normal probability density is higher in the middle compared to its two tails. Other distributions need not have this feature. There is even variation among the distributions that we call "normal." For example, some normal distributions are more spread out than the one shown in [link] (their tails begin to hit the X axis further from the middle of the curve --for example, at 10 and 90 if drawn in place of [link]). Others are less spread out (their tails might approach the X axis at 30 and 70). More information on the normal distribution can be found in a later chapter completely devoted to them.

The normal distribution shown in [link] is symmetric; if you folded it in the middle, the two sides would match perfectly. [link] shows the discrete distribution of scores on a psycholoogy test. This distribution is not symmetric: the tail in the positive direction extends further than the tail in the negative direction. A distribution with the longer tail extending in the positive direction is said to have a **positive skew**. It is also described as "skewed to the right."



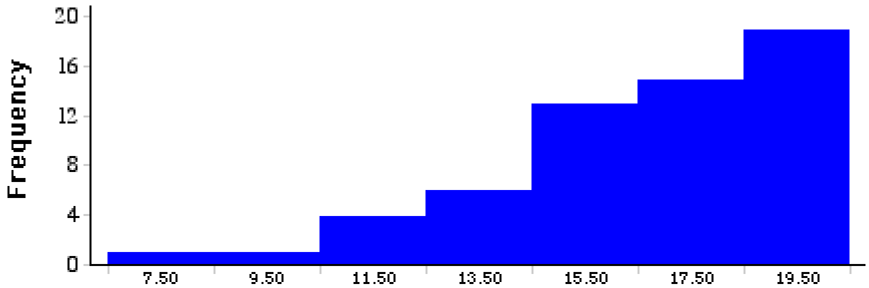
A distribution with a positive skew

[link] shows the salaries of major league baseball players in 1974 (in thousands of dollars). This distribution has an extreme positive skew.



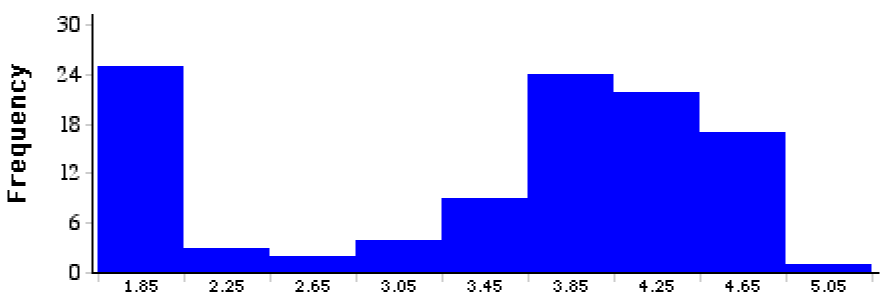
A distribution with a very large positive skew. This histogram shows the salaries of major league baseball players.

Although less common, some distributions have **negative skew**. [link] shows the scores on a 20-point problem on a statistics exam. Since the tail of the disribution extends to the left, this distribution is **skewed to the left**.



A distribution with negative skew. This histogram shows the frequencies of various scores on a 20-point question on a statistics test.

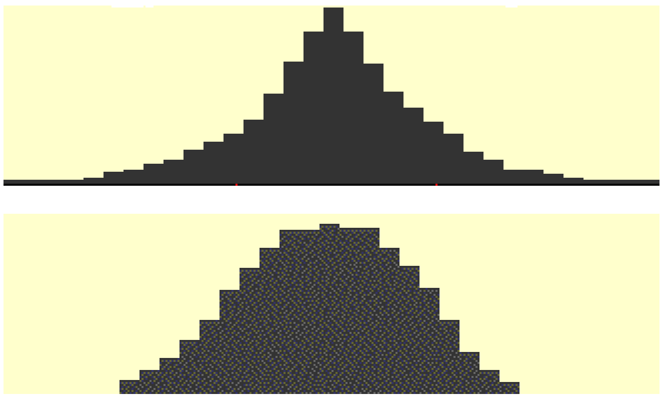
The distributions shown so far all have one distinct high point or peak. The distribution in [link] has two distinct peaks. A distribution with two peaks is called a **bimodal distribution**.



Frequencies of times between eruptions of the old faithful geyser. Notice the two distinct peaks: one at 1.85 and the other at 3.85.

Distributions also differ from each other in terms of how large or "fat" their tails are. [link] shows two distributions that differ in this respect. The upper distribution has relatively more scores in its tails; its shape is called

leptokurtic. The lower distribution has relatively fewer scores in its tails; its shape is called **platykurtic**.



Distributions differing in kurtosis. The top distribution has long tails. It is called "leptokurtic." The bottom distribution has short tails. It is called "platykurtic."

Scientific Notation

Learning Objectives

By the end of this lesson, you will be able to

- Read and understand scientific notation.
- Convert between decimal notation and scientific notation.
- Use scientific notation on a calculator.

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on.

Consider the numbers 4000 and 0.004. We know that 4000 means 4×1000 and 0.004 means $4 \times \frac{1}{1000}$. If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:

Equation:

$$\begin{array}{ll} 4000 & 0.004 \\ 4 \times 1000 & 4 \times \frac{1}{1000} \\ 4 \times 10^3 & 4 \times \frac{1}{10^3} \\ & 4 \times 10^{-3} \end{array}$$

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation.*

Note:

Scientific Notation

A number is expressed in **scientific notation** when it is of the form

Equation:

$$a \times 10^n$$

where $a \geq 1$ and $a < 10$ and n is an integer.

It is customary in scientific notation to use $\times$ as the multiplication sign, even though we avoid using this sign elsewhere in algebra.

Scientific notation is a useful way of writing very large or very small numbers. It is used often in the sciences to make calculations easier.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor, 4, by itself.

- The power of 10 is positive when the number is larger than 1: $4000 = 4 \times 10^3$.
- The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$.

Example:

Exercise:

Problem: Write $37{,}000$ in scientific notation.

Solution:

Step 1: Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	
Step 2: Count the number of decimal places, n, that the decimal point was moved.	3.70000 4 places
Step 3: Write the number as a product with a power of 10.	3.7×10^4
If the original number is: • greater than 1, the power of 10 will be 10^n. • between 0 and 1, the power of 10 will be $10^{-\mathrm{n}}$	
Step 4: Check.	
10^4 is 10,000 and 10,000 times 3.7 will be 37,000.	
	$37{,}000 = 3.7 \times 10^4$

Example:
Exercise:

Problem: Write in scientific notation: 96,000.

Solution:

9.6×10^4

Note:
Try It
Exercise:

Problem: Write in scientific notation: 48,300.

Solution:

4.83×10^4

Note:
Convert from decimal notation to scientific notation

Move the decimal point so that the first factor is greater than or equal to 1 but less than $10.$
Count the number of decimal places, $n,$ that the decimal point was moved.
Write the number as a product with a power of $10.$

- If the original number is:
- greater than 1, the power of 10 will be 10^n.
- between 0 and 1, the power of 10 will be 10^{-n}.

Check.

Example:
Exercise:

Problem: Write in scientific notation: 0.0052.

Solution:

	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}

$$5.2 \times 10^{-3}$$
$$5.2 \times \frac{1}{10^3}$$
$$5.2 \times \frac{1}{1000}$$
$$5.2 \times 0.001$$
$$0.0052$$
$$0.0052 = 5.2 \times 10^{-3}$$

Example:
Exercise:

Problem: Write in scientific notation: 0.0078.

Solution:

7.8×10^{-3}

Note:
Try It
Exercise:

Problem: Write in scientific notation: 0.0129.

Solution:

1.29×10^{-2}

Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.
Equation:

$$\begin{array}{ll} 9.12 \times 10^{4} & 9.12 \times 10^{-4} \\ 9.12 \times 10{,}000 & 9.12 \times 0.0001 \\ 91{,}200 & 0.000912 \end{array}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$$9.12 \times 10^{4} = 91{,}200 \qquad\qquad 9.12 \times 10^{-4} = 0.000912$$

$$9.12\_\,\_\times 10^{4} = 91{,}200 \qquad\qquad \_\,\_\,\_9.12 \times 10^{-4} = 0.000912$$

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

Example:
Exercise:

Problem: Convert to decimal form: 6.2×10^{3}.

Solution:

Step 1: Determine the exponent, n, on the factor 10.	6×10^{3}
Step 2: Move the decimal point n places, adding zeros if needed.	6.200,
• If the exponent is positive, move the decimal point n places to the right. • If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.	6,200

Step 3: Check to see if your answer makes sense.	
10^3 is 1000 and 1000 times 6.2 will be 6,200. $6.2 \times 10^3 = 6{,}200$	

Example:
Exercise:

Problem: Convert to decimal form: 1.3×10^3.

Solution:

1,300

Note:
Try It
Exercise:

Problem: Convert to decimal form: 9.25×10^4.

Solution:

92,500

Note:
Convert scientific notation to decimal form

Determine the exponent, n, on the factor 10.

Move the decimal	n places, adding zeros if needed.	• If the exponent is positive, move the decimal point n places to the right. • If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.
Check.		

Example:
Exercise:

Problem: Convert to decimal form: 8.9×10^{-2}.

Solution:

	8.9×10^{-2}
Determine the exponent n, on the factor 10.	The exponent is −2.
Move the decimal point 2 places to the left.	$\_8.9$
Add zeros as needed for placeholders.	0.089
	$8.9 \times 10^{-2} = 0.089$
The Check is left to you.	

Example:
Exercise:

Problem: Convert to decimal form: 1.2×10^{-4}.

Solution:

0.00012

Note:
Try It
Exercise:

Problem: Convert to decimal form: 7.5×10^{-2}.

Solution:

0.075

Scientific Notation on a Scientific Calculator

It's important that you know how to enter numbers into a calculator that are given in scientific notation, without having to convert it to decimal notation first, as well as how to read an answer that the calculator gives in scientific notation.

On a scientific calculator, there is a button specifically meant for scientific notation. This button is either an Exp or an EE button. When you press the

button an E will show on the screen (or × 10^) and the E represents × 10^, where the ^ symbol means "raised to the". You would then follow that with the exponent.

Try with your calculator: Enter in 7.15×10^3 as 7.15 Exp (or EE) 3, hit enter or the = sign, and your calculator will show 7150, which is the decimal notation.

Now enter in 7.15×10^{15} as 7.15 Exp (or EE) 15, hit enter or the = sign, and your calculator should show 7.15×10^{15} or 7.15E15. This is because this number has too many digits to show on the screen in decimal notation.

Example:
Exercise:

Problem: Multiply $(1.5 \times 10^{-7})(8.2 \times 10^3)$ using your calculator.

Solution:

This would be entered into the calculator as (1.5 Exp (or EE) −7)(8.2 Exp (or EE) 3). The calculator should give 0.00123 as the answer.

Note:
Try It
Exercise:

Problem: Multiply $(9.9 \times 10^5)(9.9 \times 10^{20})$ using your calculator.

Solution:

Your calculator may show you 9.801 E 26, which means 9.801×10^{26}.

Example:
Exercise:

Problem: Use the calculator to divide: $\frac{2.04 \times 10^5}{3.4 \times 10^{-6}}$

Solution:

This is entered as 2.04 Exp 5 ÷ 3.4 Exp -6. The answer is 6×10^{10}.

Summary

- **Convert from Decimal Notation to Scientific Notation:** To convert a decimal to scientific notation:

  Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
  Count the number of decimal places, n, that the decimal point was moved.
  Write the number as a product with a power of 10.
    - If the original number is greater than 1, the power of 10 will be 10^n.
    - If the original number is between 0 and 1, the power of 10 will be 10^n.

  Check.

- **Convert Scientific Notation to Decimal Form:** To convert scientific notation to decimal form:

  Determine the exponent, n, on the factor 10.
  Move the decimal n places, adding zeros if needed.
    - If the exponent is positive, move the decimal point n places to the right.
    - If the exponent is negative, move the decimal point $|n|$ places to the left.

Check.

Homework

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

Exercise:

Problem: 280,000

Solution:

2.8×10^5

Exercise:

Problem: 45,000

Exercise:

Problem: 1,290,000

Solution:

1.29×10^6

Exercise:

Problem: 8,750,000

Exercise:

Problem: 0.041

Solution:

4.1×10^{-2}

Exercise:

Problem: 0.036

Exercise:

Problem: 0.0000103

Solution:

1.03×10^{-5}

Exercise:

Problem: 0.00000924

Exercise:

Problem:

The population of the world on July 4, 2010 was more than 6,850,000,000.

Solution:

6.85×10^{9}

Exercise:

Problem:

The population of the United States on July 4, 2010 was almost 310,000,000.

Exercise:

Problem:

The probability of winning the 2010 Megamillions lottery is about 0.0000000057.

Solution:

5.7×10^{-9}

Exercise:

Problem: The average width of a human hair is 0.0018 centimeters.

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

Exercise:

Problem: 8.3×10^2

Solution:

830

Exercise:

Problem: 4.1×10^2

Exercise:

Problem: 1.6×10^{10}

Solution:

16,000,000,000

Exercise:

Problem: 5.5×10^8

Exercise:

Problem: 2.8×10^{-2}

Solution:

0.028

Exercise:

Problem: 3.5×10^{-2}

Exercise:

Problem: 6.15×10^{-8}

Solution:

0.0000000615

Exercise:

Problem: 1.93×10^{-5}

Exercise:

Problem:

At the start of 2012, the US federal budget had a deficit of more than $\$1.5 \times 10^{13}$.

Solution:

$15,000,000,000,000

Exercise:

Problem:

In 2010, the number of Facebook users each day who changed their status to ‘engaged’ was 2×10^{4}.

Exercise:

Problem: The width of a proton is 1×10^{-5} of the width of an atom.

Solution:

0.00001

Exercise:

Problem:

The concentration of carbon dioxide in the atmosphere is 3.9×10^{-4}.

Multiply and Divide Using Scientific Notation and a Scientific Calculator

In the following exercises, multiply or divide, using a scientific calculator, and write your answer in decimal form.

Exercise:

Problem: $(3 \times 10^{2})(1 \times 10^{-5})$

Solution:

0.003

Exercise:

Problem: $(2 \times 10^{5})(2 \times 10^{-9})$

Exercise:

Problem: $(2.1 \times 10^{-4})(3.5 \times 10^{-2})$

Solution:

0.00000735

Exercise:

Problem: $(1.6 \times 10^{-2})(5.2 \times 10^{-6})$

Exercise:

Problem: $\frac{8 \times 10^{6}}{4 \times 10^{-1}}$

Solution:

20,000,000

Exercise:

Problem: $\frac{6 \times 10^{4}}{3 \times 10^{-2}}$

Exercise:

Problem: $\frac{5 \times 10^{-3}}{1 \times 10^{-10}}$

Solution:

50,000,000

Exercise:

Problem: $\frac{7 \times 10^{-2}}{1 \times 10^{-8}}$

Averages and Probability
By the end of this section, you will be able to:

- Calculate the mean of a set of numbers
- Find the median of a set of numbers
- Find the mode of a set of numbers
- Apply the basic definition of probability

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{4+9+2}{3}$.
   If you missed this problem, review [link].
2. Simplify: $4(8)+6(3)$.
   If you missed this problem, review [link].
3. Convert $\frac{5}{2}$ to a decimal.
   If you missed this problem, review [link].

One application of decimals that arises often is finding the *average* of a set of numbers. What do you think of when you hear the word *average*? Is it your grade point average, the average rent for an apartment in your city, the batting average of a player on your favorite baseball team? The average is a typical value in a set of numerical data. Calculating an average sometimes involves working with decimal numbers. In this section, we will look at three different ways to calculate an average.

Calculate the Mean of a Set of Numbers

The mean is often called the arithmetic average. It is computed by dividing the sum of the values by the number of values. Students want to know the mean of their test scores. Climatologists report that the mean temperature has, or has not, changed. City planners are interested in the mean household size.

Suppose Ethan’s first three test scores were $85, 88, \text{and } 94$. To find the mean score, he would add them and divide by 3.
Equation:

$$\frac{85+88+94}{3}$$
$$\frac{267}{3}$$
$$89$$

His mean test score is 89 points.

Note:
The Mean
The **mean** of a set of n numbers is the arithmetic average of the numbers.
Equation:

$$\text{mean} = \frac{\text{sum of values in data set}}{n}$$

Note:
Calculate the mean of a set of numbers.

Write the formula for the mean **Equation:**

$$\text{mean} = \frac{\text{sum of values in data set}}{n}$$

Find the sum of all the values in the set. Write the sum in the numerator.
Count the number, n, of values in the set. Write this number in the denominator.
Simplify the fraction.
Check to see that the mean is reasonable. It should be greater than the least number and less than the greatest number in the set.

Example:
Exercise:

Problem: Find the mean of the numbers $8, 12, 15, 9$, and 6.

Solution:
Solution

Write the formula for the mean:	$\text{mean} = \frac{\text{sum of all the numbers}}{n}$
Write the sum of the numbers in the numerator.	$\text{mean} = \frac{8+12+15+9+6}{n}$
Count how many numbers are in the set. There are 5 numbers in the set, so $n = 5$.	$\text{mean} = \frac{8+12+15+9+6}{5}$
Add the numbers in the numerator.	$\text{mean} = \frac{50}{5}$
Then divide.	$\text{mean} = 10$
Check to see that the mean is 'typical': 10 is neither less than 6 nor greater than 15.	The mean is 10.

Note:
Exercise:

Problem: Find the mean of the numbers: $8, 9, 7, 12, 10, 5$.

Solution:

8.5

Note:
Exercise:

Problem: Find the mean of the numbers: $9, 13, 11, 7, 5$.

Solution:

9

Example:
Exercise:

Problem:

The ages of the members of a family who got together for a birthday celebration were $16, 26, 53, 56, 65, 70, 93,$ and 97 years. Find the mean age.

Solution:
Solution

Write the formula for the mean:	$\text{mean} = \frac{\text{sum of all the numbers}}{n}$
Write the sum of the numbers in the numerator.	$\text{mean} = \frac{16+26+53+56+65+70+93+97}{n}$
Count how many numbers are in the set. Call this n and write it in the denominator.	$\text{mean} = \frac{16+26+53+56+65+70+93+97}{8}$
Simplify the fraction.	$\text{mean} = \frac{476}{8}$
	$\text{mean} = 59.5$

Is 59.5 'typical'? Yes, it is neither less than 16 nor greater than 97. The mean age is 59.5 years.

Note:
Exercise:

Problem:

The ages of the four students in Ben's carpool are $25, 18, 21,$ and 22. Find the mean age of the students.

Solution:

21.5 years years

Note:
Exercise:

Problem:

Yen counted the number of emails she received last week. The numbers were $4, 9, 15, 12, 10, 12,$ and 8. Find the mean number of emails.

Solution:

10

Did you notice that in the last example, while all the numbers were whole numbers, the mean was 59.5, a number with one decimal place? It is customary to report the mean to one more decimal place than the original numbers. In the next example, all the numbers represent money, and it will make sense to report the mean in dollars and cents.

Example:
Exercise:

Problem:

For the past four months, Daisy's cell phone bills were $\$42.75, \$50.12, \$41.54, \$48.15$. Find the mean cost of Daisy's cell phone bills.

Solution:
Solution

Write the formula for the mean.	$\text{mean} = \frac{\text{sum of all the numbers}}{n}$
Count how many numbers are in the set. Call this n and write it in the denominator.	$\text{mean} = \frac{\text{sum of all the numbers}}{4}$

Write the sum of all the numbers in the numerator.	$\text{mean} = \frac{42.75+50.12+41.54+48.15}{4}$
Simplify the fraction.	$\text{mean} = \frac{182.56}{4}$
	$\text{mean} = 45.64$

Does $45.64 seem 'typical' of this set of numbers? Yes, it is neither less than $41.54 nor greater than $50.12.

The mean cost of her cell phone bill was $45.64

Note:
Exercise:

Problem:

Last week Ray recorded how much he spent for lunch each workday. He spent $6.50, $7.25, $4.90, $5.30, and $12.00. Find the mean of how much he spent each day.

Solution:

$7.19

Note:
Exercise:

Problem:

Lisa has kept the receipts from the past four trips to the gas station. The receipts show the following amounts: $34.87, $42.31, $38.04, and $43.26. Find the mean.

Solution:

$39.62

Find the Median of a Set of Numbers

When Ann, Bianca, Dora, Eve, and Francine sing together on stage, they line up in order of their heights. Their heights, in inches, are shown in [link].

Ann	Bianca	Dora	Eve	Francine

Ann	Bianca	Dora	Eve	Francine
59	60	65	68	70

Dora is in the middle of the group. Her height, $65''$, is the *median* of the girls' heights. Half of the heights are less than or equal to Dora's height, and half are greater than or equal. The median is the middle value.

Note:
Median
The **median** of a set of data values is the middle value.

- Half the data values are less than or equal to the median.
- Half the data values are greater than or equal to the median.

What if Carmen, the pianist, joins the singing group on stage? Carmen is 62 inches tall, so she fits in the height order between Bianca and Dora. Now the data set looks like this:
Equation:

$$59, 60, 62, 65, 68, 70$$

There is no single middle value. The heights of the six girls can be divided into two equal parts.

Statisticians have agreed that in cases like this the median is the mean of the two values closest to the middle. So the median is the mean of 62 and 65, $\frac{62+65}{2}$. The median height is 63.5 inches.

Notice that when the number of girls was 5, the median was the third height, but when the number of girls was 6, the median was the mean of the third and fourth heights. In general, when the number of values is odd, the median will be the one value in the middle, but when the number is even, the median is the mean of the two middle values.

Note:
Find the median of a set of numbers.

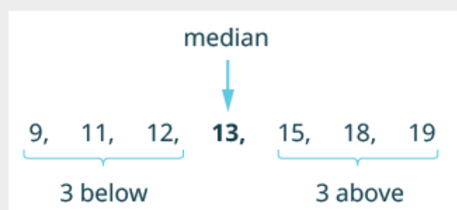
List the numbers from smallest to largest.
Count how many numbers are in the set. Call this n.
Is n odd or even?

- If n is an odd number, the median is the middle value.
- If n is an even number, the median is the mean of the two middle values.

Example:
Exercise:

Problem: Find the median of $12, 13, 19, 9, 11, 15$, and 18.

Solution:
Solution

List the numbers in order from smallest to largest.	9, 11, 12, 13, 15, 18, 19
Count how many numbers are in the set. Call this n .	$n = 7$
Is n odd or even?	odd
The median is the middle value.	
The middle is the number in the 4th position.	So the median of the data is 13.

Note:
Exercise:

Problem: Find the median of the data set: $43, 38, 51, 40, 46$.

Solution:

43

Note:
Exercise:

Problem: Find the median of the data set: $15, 35, 20, 45, 50, 25, 30$.

Solution:

30

Example:
Exercise:

Problem: Kristen received the following scores on her weekly math quizzes:

$83, 79, 85, 86, 92, 100, 76, 90, 88, \text{and } 64.$ Find her median score.

Solution:
Solution

Find the median of 83, 79, 85, 86, 92, 100, 76, 90, 88, and 64.	
List the numbers in order from smallest to largest.	64, 76, 79, 83, 85, 86, 88, 90, 92, 100
Count the number of data values in the set. Call this n.	$n = 10$
Is n odd or even?	even
The median is the mean of the two middle values, the 5th and 6th numbers.	$\underbrace{64, 76, 79, 83, \mathbf{85},}_{\text{5 numbers}} \ \underbrace{\mathbf{86,}\ 88, 90, 92, 100}_{\text{5 numbers}}$
Find the mean of 85 and 86.	$\text{mean} = \frac{85+86}{2}$
	$\text{mean} = 85.5$
	Kristen's median score is 85.5.

Note:
Exercise:

Problem: Find the median of the data set: $8, 7, 5, 10, 9, 12$.

Solution:

8.5

Note:
Exercise:

Problem: Find the median of the data set: $21, 25, 19, 17, 22, 18, 20, 24$.

Solution:

20.5

Identify the Mode of a Set of Numbers

The *average* is one number in a set of numbers that is somehow typical of the whole set of numbers. The mean and median are both often called the average. Yes, it can be confusing when the word average refers to two different numbers, the mean and the median! In fact, there is a third number that is also an average. This average is the **mode**. The mode of a set of numbers is the number that occurs the most. The **frequency**, is the number of times a number occurs. So the mode of a set of numbers is the number with the highest frequency.

Note:
Mode
The **mode** of a set of numbers is the number with the highest frequency.

Suppose Jolene kept track of the number of miles she ran since the start of the month, as shown in [link].

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
				1 2 mi New Year's day	2	3 15 mi
4 8 mi	5	6 3 mi	7 8 mi	8	9 5 mi	10 8 mi
11	12	13	14	15	16	17

If we list the numbers in order it is easier to identify the one with the highest frequency.
Equation:

$$2, 3, 5, 8, 8, 8, 15$$

Jolene ran 8 miles three times, and every other distance is listed only once. So the mode of the data is 8 miles.

Note:
Identify the mode of a set of numbers.

List the data values in numerical order.
Count the number of times each value appears.
The mode is the value with the highest frequency.

Example:
Exercise:

Problem: The ages of students in a college math class are listed below. Identify the mode.

$$18, 18, 18, 18, 19, 19, 19, 20, 20, 20, 20, 20, 20, 20, 21, 21, 22, 22, 22, 22, 22, 23, 24, 24, 25, 29, 30, 40, 44$$

Solution:
Solution

The ages are already listed in order. We will make a table of frequencies to help identify the age with the highest frequency.

Age	18	19	20	21	22	23	24	25	29	30	40	44
Frequency	4	3	7	2	5	1	2	1	1	1	1	1

Now look for the highest frequency. The highest frequency is 7, which corresponds to the age 20. So the mode of the ages in this class is 20 years.

Note:
Exercise:

Problem:

The number of sick days employees used last year: 3, 6, 2, 3, 7, 5, 6, 2, 4, 2. Identify the mode.

Solution:

2

Note:
Exercise:

Problem:

The number of handbags owned by women in a book club: $5, 6, 3, 1, 5, 8, 1, 5, 8, 5$. Identify the mode.

Solution:

5

Example:
Exercise:

Problem: The data lists the heights (in inches) of students in a statistics class. Identify the mode.

56	61	63	64	65	66	67	67
60	62	63	64	65	66	67	70
60	63	63	64	66	66	67	74
61	63	64	65	66	67	67	

Solution:
Solution

List each number with its frequency.

Number	56	60	61	62	63	64	65	66	67	70	74
Frequency	1	2	2	1	5	4	3	5	6	1	1

Now look for the highest frequency. The highest frequency is 6, which corresponds to the height 67 inches. So the mode of this set of heights is 67 inches.

Note:
Exercise:

Problem:

The ages of the students in a statistics class are listed here: 19, 20, 23, 23, 38, 21, 19, 21, 19, 21, 20, 43, 20, 23, 17, 21, 21, 20, 29, 18, 28. What is the mode?

Solution:

21

Note:
Exercise:

Problem:

Students listed the number of members in their household as follows: $6, 2, 5, 6, 3, 7, 5, 6, 5, 3, 4, 4, 5, 7, 6, 4, 5, 2, 1, 5$. What is the mode?

Solution:

5

Some data sets do not have a mode because no value appears more than any other. And some data sets have more than one mode. In a given set, if two or more data values have the same highest frequency, we say they are all modes.

Use the Basic Definition of Probability

The probability of an event tells us how likely that event is to occur. We usually write probabilities as fractions or decimals.

For example, picture a fruit bowl that contains five pieces of fruit - three bananas and two apples.

If you want to choose one piece of fruit to eat for a snack and don't care what it is, there is a $\frac{3}{5}$ probability you will choose a banana, because there are three bananas out of the total of five pieces of fruit. The probability of an event is the number of favorable outcomes divided by the total number of outcomes.

$$\text{Probability of an event} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

$$\text{Probability of choosing a banana} = \frac{3}{5}$$

There are 3 bananas.
There are 5 pieces of fruit.

Note:
Probability
The **probability** of an event is the number of favorable outcomes divided by the total number of outcomes possible.
Equation:

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Converting the fraction $\frac{3}{5}$ to a decimal, we would say there is a 0.6 probability of choosing a banana.
Equation:

$$\text{Probability of choosing a banana} = \frac{3}{5}$$
$$\text{Probability of choosing a banana} = 0.6$$

This basic definition of probability assumes that all the outcomes are equally likely to occur. If you study probabilities in a later math class, you'll learn about several other ways to calculate probabilities.

Example:
Exercise:

Problem:

The ski club is holding a raffle to raise money. They sold 100 tickets. All of the tickets are placed in a jar. One ticket will be pulled out of the jar at random, and the winner will receive a prize. Cherie bought one raffle ticket.

ⓐ Find the probability she will win the prize.

ⓑ Convert the fraction to a decimal.

Solution:
Solution

ⓐ	
What are you asked to find?	The probability Cherie wins the prize.
What is the number of favorable outcomes?	1, because Cherie has 1 ticket.
Use the definition of probability.	$\text{Probability of an event} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$
Substitute into the numerator and denominator.	$\text{Probability Cherie wins} = \frac{1}{100}$

ⓑ	
Convert the fraction to a decimal.	
Write the probability as a fraction.	$\text{Probability} = \frac{1}{100}$

Convert the fraction to a decimal.	$\text{Probability} = 0.01$

Note:
Exercise:

Problem:

Ignaly is attending a fashion show where the guests are seated at tables of ten. One guest from each table will be selected at random to receive a door prize. ⓐ Find the probability Ignaly will win the door prize for her table. ⓑ Convert the fraction to a decimal.

Solution:

ⓐ $\frac{1}{10}$
ⓑ 0.1

Note:
Exercise:

Problem:

Hoang is among 20 people available to sit on a jury. One person will be chosen at random from the 20. ⓐ Find the probability Hoang will be chosen. ⓑ Convert the fraction to a decimal.

Solution:

ⓐ $\frac{1}{20}$
ⓑ 0.05

Example:
Exercise:

Problem:

Three women and five men interviewed for a job. One of the candidates will be offered the job.

ⓐ Find the probability the job is offered to a woman.

ⓑ Convert the fraction to a decimal.

Solution:
Solution

ⓐ	
What are you asked to find?	The probability the job is offered to a woman.
What is the number of favorable outcomes?	3, because there are three women.
What are the total number of outcomes?	8, because 8 people interviewed.
Use the definition of probability.	$\text{Probability of an event} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$
Substitute into the numerator and denominator.	$\text{Probability} = \frac{3}{8}$

ⓑ	
Convert the fraction to a decimal.	
Write the probability as a fraction.	$\text{Probability} = \frac{3}{8}$
Convert the fraction to a decimal.	$\text{Probability} = 0.375$

Note:
Exercise:

Problem:

A bowl of Halloween candy contains 5 chocolate candies and 3 lemon candies. Tanya will choose one piece of candy at random. ⓐ Find the probability Tanya will choose a chocolate candy. ⓑ Convert the fraction to a decimal.

Solution:

ⓐ $\frac{5}{8}$
ⓑ 0.625

Note:
Exercise:

Problem:

Dan has 2 pairs of black socks and 6 pairs of blue socks. He will choose one pair at random to wear tomorrow. ⓐ Find the probability Dan will choose a pair of black socks ⓑ Convert the fraction to a decimal.

Solution:

ⓐ $\frac{2}{8}$
ⓑ 0.25

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Mean, Median, and Mode
- Find the Mean of a Data Set
- Find the Median of a Data Set
- Find the Mode of a Data Set

Key Concepts

- **Calculate the mean of a set of numbers.**

  Write the formula for the mean $\text{mean} = \frac{\text{sum of values in data set}}{n}$
  Find the sum of all the values in the set. Write the sum in the numerator.
  Count the number, *n*, of values in the set. Write this number in the denominator.
  Simplify the fraction.
  Check to see that the mean is reasonable. It should be greater than the least number and less than the greatest number in the set.

- **Find the median of a set of numbers.**

  List the numbers from least to greatest.
  Count how many numbers are in the set. Call this n.

Is n odd or even?	If n is an odd number, the median is the middle value.	If n is an even number, the median is the mean of the two middle values

- **Identify the mode of a set of numbers.**

  List the data values in numerical order.
  Count the number of times each value appears.
  The mode is the value with the highest frequency.

Practice Makes Perfect

Calculate the Mean of a Set of Numbers

In the following exercises, find the mean.

Exercise:

Problem: 3, 8, 2, 2, 5

Solution:

4

Exercise:

Problem: 6, 1, 9, 3, 4, 7

Exercise:

Problem: 65, 13, 48, 32, 19, 33

Solution:

35

Exercise:

Problem: 34, 45, 29, 61, and 41

Exercise:

Problem: 202, 241, 265, 274

Solution:

245.5

Exercise:

Problem: 525, 532, 558, 574

Exercise:

Problem: 12.45, 12.99, 10.50, 11.25, 9.99, 12.72

Solution:

11.65

Exercise:

Problem: 28.8, 32.9, 32.5, 27.9, 30.4, 32.5, 31.6, 32.7

Exercise:

Problem:

Four girls leaving a mall were asked how much money they had just spent. The amounts were $0, $14.95, $35.25, and $25.16. Find the mean amount of money spent.

Solution:

$18.84

Exercise:

Problem:

Juan bought 5 shirts to wear to his new job. The costs of the shirts were $32.95, $38.50, $30.00, $17.45, and $24.25. Find the mean cost.

Exercise:

Problem:

The number of minutes it took Jim to ride his bike to school for each of the past six days was 21, 18, 16, 19, 24, and 19. Find the mean number of minutes.

Solution:

19.5 minutes

Exercise:

Problem:

Norris bought six books for his classes this semester. The costs of the books were $74.28, $120.95, $52.40, $10.59, $35.89, and $59.24. Find the mean cost.

Exercise:

Problem:

The top eight hitters in a softball league have batting averages of .373, .360, .321, .321, .320, .312, .311, and .311. Find the mean of the batting averages. Round your answer to the nearest thousandth.

Solution:

0.329

Exercise:

Problem:

The monthly snowfall at a ski resort over a six-month period was 60.3, 79.7, 50.9, 28.0, 47.4, and 46.1 inches. Find the mean snowfall.

Find the Median of a Set of Numbers

In the following exercises, find the median.

Exercise:

Problem: 24, 19, 18, 29, 21

Solution:

21

Exercise:

Problem: 48, 51, 46, 42, 50

Exercise:

Problem: 65, 56, 35, 34, 44, 39, 55, 52, 45

Solution:

45

Exercise:

Problem: 121, 115, 135, 109, 136, 147, 127, 119, 110

Exercise:

Problem: 4, 8, 1, 5, 14, 3, 1, 12

Solution:

4.5

Exercise:

Problem: 3, 9, 2, 6, 20, 3, 3, 10

Exercise:

Problem: 99.2, 101.9, 98.6, 99.5, 100.8, 99.8

Solution:

99.65

Exercise:

Problem: 28.8, 32.9, 32.5, 27.9, 30.4, 32.5, 31.6, 32.7

Exercise:

Problem:

Last week Ray recorded how much he spent for lunch each workday. He spent $6.50, $7.25, $4.90, $5.30, and $12.00. Find the median.

Solution:

$6.50

Exercise:

Problem:

Michaela is in charge of 6 two-year olds at a daycare center. Their ages, in months, are 25, 24, 28, 32, 29, and 31. Find the median age.

Exercise:

Problem:

Brian is teaching a swim class for 6 three-year olds. Their ages, in months, are $38, 41, 45, 36, 40, \text{and } 42.$ Find the median age.

Solution:

40.5 months

Exercise:

Problem:

Sal recorded the amount he spent for gas each week for the past 8 weeks. The amounts were $\$38.65$, $\$32.18$, $\$40.23$, $\$51.50$, $\$43.68$, $\$30.96$, $\$41.37$, and $\$44.72.$ Find the median amount.

Identify the Mode of a Set of Numbers

In the following exercises, identify the mode.

Exercise:

Problem: 2, 5, 1, 5, 2, 1, 2, 3, 2, 3, 1

Solution:

2

Exercise:

Problem: 8, 5, 1, 3, 7, 1, 1, 7, 1, 8, 7

Exercise:

Problem: 18, 22, 17, 20, 19, 20, 22, 19, 29, 18, 23, 25, 22, 24, 23, 22, 18, 20, 22, 20

Solution:

22

Exercise:

Problem: 42, 28, 32, 35, 24, 32, 48, 32, 32, 24, 35, 28, 30, 35, 45, 32, 28, 32, 42, 42, 30

Exercise:

Problem: The number of children per house on one block: 1, 4, 2, 3, 3, 2, 6, 2, 4, 2, 0, 3, 0.

Solution:

2 children

Exercise:

Problem: The number of movies watched each month last year: 2, 0, 3, 0, 0, 8, 6, 5, 0, 1, 2, 3.

Exercise:

Problem:

The number of units being taken by students in one class: $12, 5, 11, 10, 10, 11, 5, 11, 11, 11, 10, 12$.

Solution:

11 units

Exercise:

Problem:

The number of hours of sleep per night for the past two weeks: $8, 5, 7, 8, 8, 6, 6, 6, 6, 9, 7, 8, 8, 8$.

Use the Basic Definition of Probability

In the following exercises, express the probability as both a fraction and a decimal. (Round to three decimal places, if necessary.)

Exercise:

Problem:

Josue is in a book club with 20 members. One member is chosen at random each month to select the next month's book. Find the probability that Josue will be chosen next month.

Solution:

$\frac{1}{20}, 0.05$

Exercise:

Problem:

Jessica is one of eight kindergarten teachers at Mandela Elementary School. One of the kindergarten teachers will be selected at random to attend a summer workshop. Find the probability that Jessica will be selected.

Exercise:

Problem:

There are 24 people who work in Dane's department. Next week, one person will be selected at random to bring in doughnuts. Find the probability that Dane will be selected. Round your answer to the nearest thousandth.

Solution:

$\frac{1}{24}, 0.041\overline{6} \approx 0.042$

Exercise:

Problem:

Monica has two strawberry yogurts and six banana yogurts in her refrigerator. She will choose one yogurt at random to take to work. Find the probability Monica will choose a strawberry yogurt.

Exercise:

Problem:

Michel has four rock CDs and six country CDs in his car. He will pick one CD to play on his way to work. Find the probability Michel will pick a rock CD.

Solution:

$\frac{4}{10}, 0.4$

Exercise:

Problem:

Noah is planning his summer camping trip. He can't decide among six campgrounds at the beach and twelve campgrounds in the mountains, so he will choose one campground at random. Find the probability that Noah will choose a campground at the beach.

Exercise:

Problem:

Donovan is considering transferring to a 4-year college. He is considering 10 out-of state colleges and 4 colleges in his state. He will choose one college at random to visit during spring break. Find the probability that Donovan will choose an out-of-state college.

Solution:

$\frac{10}{14}, 0.\overline{714285} \approx 0.714$

Exercise:

Problem:

There are $258{,}890{,}850$ number combinations possible in the Mega Millions lottery. One winning jackpot ticket will be chosen at random. Brent chooses his favorite number combination and buys one ticket. Find the probability Brent will win the jackpot. Round the decimal to the first digit that is not zero, then write the name of the decimal.

Everyday Math

Exercise:

Problem:

Joaquin gets paid every Friday. His paychecks for the past 8 Fridays were $\$315$, $\$236.25$, $\$236.25$, $\$236.25$ $\$315$, $\$315$, $\$236.25$, $\$393.75$. Find the ⓐ mean, ⓑ median, and ⓒ mode.

Solution:

ⓐ $285.47
ⓑ $275.63
ⓒ $236.25

Exercise:

Problem:

The cash register receipts each day last week at a coffee shop were $\$1,845$, $\$1,520$, $\$1,438$, $\$1,682$, $\$1,850$, $\$2,721$, $\$2,539$. Find the ⓐ mean, ⓑ median, and ⓒ mode.

Writing Exercises

Exercise:

Problem:

Explain in your own words the difference between the mean, median, and mode of a set of numbers.

Solution:

Answers will vary.

Exercise:

Problem:

Make an example of probability that relates to your life. Write your answer as a fraction and explain what the numerator and denominator represent.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
calculate the mean of a set of numbers.			
find the median of a set of numbers.			
find the mode of a set of numbers.			
use the basic definition of probability.			

ⓑ After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

Glossary

mean
: The mean of a set of n numbers is the arithmetic average of the numbers. The formula is $\text{mean} = \frac{\text{sum of values in data set}}{n}$

median
: The median of a set of data values is the middle value.

- Half the data values are less than or equal to the median.

- Half the data values are greater than or equal to the median.

mode
: The mode of a set of numbers is the number with the highest frequency.

Rounding Decimals
This module discusses how to round decimals.

Learning Objectives

By the end of this lesson, you should be able to:

- Round a decimal number to a specified place value
- Round a decimal number to a specified number of decimal places

Rounding Decimal Numbers

We first considered the concept of rounding numbers in [link] where our concern with rounding was related to whole numbers only. With a few minor changes, we can apply the same rules of rounding to decimals.

To round a decimal to a particular position:

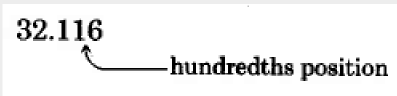
1. Mark the position of the round-off digit (with an arrow or check).
2. Note whether the digit to the immediate right of the marked digit is
   a. *less than 5*. If so, leave the round-off digit unchanged.
   b. *5 or greater*. If so, add 1 to the round-off digit.
3. If the round-off digit is
   a. to the right of the decimal point, eliminate all the digits to its right.
   b. to the left of the decimal point, replace all the digits between it and the decimal point with zeros and eliminate the decimal point and all the decimal digits.

Examples

Round each decimal to the specified place value. (The numbers in parentheses indicate which step is being used.)

Example:
Round 32.116 to the nearest hundredth.

- **1**

- **2b** The digit immediately to the right is 6, and 6 > 5, so we add 1 to the round-off digit:

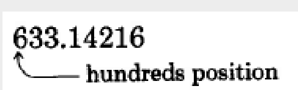
  $1 + 1 = 2$
- **3a** The round-off digit is to the right of the decimal point, so we eliminate all digits to its right.

  32.12

The number 32.116 rounded to the nearest hundredth is 32.12.

Example:
Round 633.14216 to the nearest hundred.

- **1**

- **2a** The digit immediately to the right is 3, and 3 < 5 so we leave the round-off digit unchanged.
- **3b** The round-off digit is to the left of 0, so we replace all the digits between it and the decimal point with zeros and eliminate the decimal point and all the decimal digits.

  600

The number 633.14216 rounded to the nearest hundred is 600.

Example:
1,729.63 rounded to the nearest ten is 1,730.

Example:
1.0144 rounded to the nearest tenth is 1.0.

Example:
60.98 rounded to the nearest one is 61.

Sometimes we hear a phrase such as "round to three decimal places." This phrase means that the round-off digit is the third decimal digit (the digit in the thousandths position).

Example:
67.129 rounded to the second decimal place is 67.13.

Example:
67.129558 rounded to 3 decimal places is 67.130.

Practice

Round each decimal to the specified place value or number of decimal places.

Note:

Try It

Exercise:

Problem: 4.816 to the nearest hundredth.

Solution:

4.82

Note:

Try It

Exercise:

Problem: 0.35928 to the nearest ten thousandths.

Solution:

0.3593

Note:

Try It

Exercise:

Problem: 82.1 to the nearest one.

Solution:

82

Note:

Try It
Exercise:

Problem: 753.98 to the nearest hundred.

Solution:

800

Note:
Try It
Exercise:

Problem: Round 43.99446 to three decimal places.

Solution:

43.994

Note:
Try It
Exercise:

Problem: Round 105.019997 to four decimal places.

Solution:

105.0200

Homework

For the first 10 problems, complete the chart by rounding each decimal to the indicated place value.

Exercise:

Problem: 20.01071

Tenth	Hundredth	Thousandth	Ten Thousandth

Solution:

Tenth	Hundredth	Thousandth	Ten Thousandth
20.0	20.01	20.011	20.0107

Exercise:

Problem: 3.52612

Tenth	Hundredth	Thousandth	Ten Thousandth
	3.53		

Exercise:

Problem: 531.21878

Tenth	Hundredth	Thousandth	Ten Thousandth

Solution:

Tenth	Hundredth	Thousandth	Ten Thousandth
531.2	531.22	531.219	531.2188

Exercise:

Problem: 36.109053

Tenth	Hundredth	Thousandth	Ten Thousandth
36.1			

Exercise:

Problem: 1.999994

Tenth	Hundredth	Thousandth	Ten Thousandth

Solution:

Tenth	Hundredth	Thousandth	Ten Thousandth
2.0	2.00	2.000	2.0000

Exercise:

Problem: 7.4141998

Tenth	Hundredth	Thousandth	Ten Thousandth
		7.414	

Exercise:

Problem: 0.000007

Tenth	Hundredth	Thousandth	Ten Thousandth

Solution:

Tenth	Hundredth	Thousandth	Ten Thousandth
0.0	0.00	0.000	0.0000

Exercise:

Problem: 0.00008

Tenth	Hundredth	Thousandth	Ten Thousandth
			0.0001

Exercise:

Problem: 9.19191919

Tenth	Hundredth	Thousandth	Ten Thousandth

Solution:

Tenth	Hundredth	Thousandth	Ten Thousandth
9.2	9.19	9.192	9.1919

Exercise:

Problem: 0.0876543

Tenth	Hundredth	Thousandth	Ten Thousandth

Calculator Problems

For the following 5 problems, round 18.4168095 to the indicated number of decimal places.

Exercise:

Problem: 3 decimal places.

Solution:

18.417

Exercise:

Problem: 1 decimal place.

Exercise:

Problem: 5 decimal places.

Solution:

18.41681

Exercise:

Problem: 6 decimal places.

Exercise:

Problem: 2 decimal places.

Solution:

18.42

Calculator Problems

For the following problems, perform each division using a calculator.

Exercise:

Problem: $4 \div 3$ and round to 2 decimal places.

Exercise:

Problem: $1 \div 8$ and round to 1 decimal place.

Solution:

0.1

Exercise:

Problem: $1 \div 27$ and round to 6 decimal places.

Exercise:

Problem: $51 \div 61$ and round to 5 decimal places.

Solution:

0.83607

Exercise:

Problem: $3 \div 16$ and round to 3 decimal places.

Exercise:

Problem: $16 \div 3$ and round to 3 decimal places.

Solution:

5.333

Exercise:

Problem: $26 \div 7$ and round to 5 decimal places.

[Optional Topic] Rounding with Precision and Significant Figures

A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The “known masses” are typically metal cylinders of standard mass such as 1 gram, 10 grams, and 100 grams. (credit: Serge Melki)

Learning Objectives

By the end of this section, you will be able to:

- Round answers from adding or subtracting measurements based on the least precise place value.
- Determine the appropriate number of significant figures to use in multiplication and division calculations.

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. The fields of health and medicine are also based heavily on measurements. **Accuracy** is how close a measurement is to the correct value for that measurement. For example, let's say that you are measuring the length of standard computer paper. The packaging in which you purchased the paper states that it is 11.0 inches long. You measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the correct value of 11.0 inches. In contrast, if you had obtained a measurement of 12 inches, your measurement would not be very accurate.

The **precision** of a measurement system has to do with place value. For example, each of the measurements of the piece of paper are precise to the tenths place.

Here are some examples of measurements and their precision as shown by place values:

- 13.00 is precise to the hundredths place.
- 0.140 is precise to the thousandths place.
- 3400 is precise to the hundreds place.
- 239,000 is precise to the thousands place.
- 14 is precise to the ones place.

If there are no digits past the decimal, then the precision of the measurement is to the last place value with a nonzero digit, as you read the number from left to right. If there are digits to the right of the decimal, then the precision of the measurement is to the place value of the very last digit, as you read the number from left to right, even if the last digit is a zero.

Can you see how adding zeros past the decimal without reason can make a measurement more precise and dropping zeros will make a measurement less precise? If a measurement of 13.00, which is precise to the hundredths place, was written down as 13 or 13.0 or 13.000 then this would be wrong since it needs to have its last zero in the hundredths place exactly.

Note: Measurements are always rounded estimates because of the inaccuracy of the tool used. The idea of being an estimate must be reflected in the way the results are recorded.

The result of mathematical operations, performed with approximate numbers from measurements, is also an approximate number and must be rounded according to either the accuracy or the precision of the original numbers.

The output cannot be more accurate or precise than the input. This is the main idea! Again, **the output cannot be more accurate or precise than the input.**

Precision when Adding and Subtracting Measurements

When adding and subtracting measurements, the result must be rounded to the **least precise place value** of the measurements involved. The key is that **the result cannot be any more precise than the least precise measurement given.**

When given two or more measurements to add (or subtract), first add as normal. In other words, **do not round the numbers before you add**. Then, after adding (or subtracting), round the result to the least precise place value of the given numbers.

In our example, such factors contributing to the uncertainty could be the following: the smallest division on the ruler is 0.1 in., the person using the ruler has bad eyesight, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be based on a careful consideration of all the factors that might contribute and their possible effects.

Example:
Exercise:

Problem: Add: 15,800 mi. + 11,000 mi.

Solution:

We first add the numbers to get 26,800 mi.

Next we note that 15,800 is precise to the hundreds place and 11,000 is precise to the thousands place.

The thousands place is less precise than the hundreds place, so we round the result to the thousands place.

Thus, the final answer is 27,000 mi. (Notice how the units of miles is included.)

Note:
Steps for adding and subtracting measurements with precision

Make sure the measurements are in the same units. (We can’t add km to m, for example)
Add or subtract the numbers as they are given to you.
Round the result to the least precise place value.
Label the result with the correct units. Don't forget this very important step!

Example:
Exercise:

Problem: Add: 58.03 m + 11 m

Solution:

First add the numbers to get 69.03 m.

Then, note that 58.03 is precise to the hundredths place and 11 is precise to the ones place.

The hundredths place is more precise then the ones place, so we need to round the result to the ones place.

Thus, the final answer is 69 m.

Note:
Try It
Exercise:

Problem:

Add 24.3 ft. and 16 ft. Round the result to the least precise place value.

Solution:

The answer is 40 ft. Note that this result is precise to the tens place due to the zero in the ones place. However, the key is that the result cannot be any more precise than the least precise place value of the measurements given, which indeed is the case here.

Example:
Exercise:

Problem: Subtract 344 cm from 48 m.

Solution:

First, we must convert these into the same units. We can either convert the centimeters to meters or convert the meters to centimeters.

If we convert the meters to centimeters, then this will give us a subtraction problem of 4,800 cm - 344 cm.

Next, we do the subtraction, which gives us 4,456 cm.

Note that 4,800 is precise to the hundreds place and 344 is precise to the ones place. Since the hundreds place is less precise than the ones place, we round our answer to the hundreds place.

The final answer is 4,500 cm.

Precision when Multiplying and Dividing Measurements, Using Significant Figures

Because measurements are approximations they are rounded estimates. When multiplying or dividing by a rounded number, the result must be rounded correctly to reflect the estimation. This is done by considering the significant digits of each number.

For multiplication and division: The result should have the same number of significant figures as the measurement having the least significant figures entering into the calculation.

When we express measured values, we can only list as many digits as we initially measured with our measuring tool. For example, if you use a standard ruler to measure the length of a stick, you may measure it to be $36.7\ \mathrm{cm}$. You could not express this value as $36.71\ \mathrm{cm}$ because your measuring tool was not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices that the stick length seems to be somewhere in between $36.6\ \mathrm{cm}$ and

$36.7\ \mathrm{cm}$, and he or she must estimate the value of the last digit. Using the method of **significant figures**, the rule is that *the last digit written down in a measurement is the first digit with some uncertainty*. In order to determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value $36.7\ \mathrm{cm}$ has three digits, or significant figures. Significant figures indicate the precision of a measuring tool that was used to measure a value.

Zeros

Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant, because they are only placeholders that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placeholders, but are significant — this number has five significant figures. The zeros in 1300 may or may not be significant depending on the style of writing numbers. They could mean the number is known to the last digit, or the zeros could be placeholders. So 1300 could have two, three, or four significant figures. (To avoid this ambiguity, write 1300 in scientific notation.) ***Zeros are significant except when they serve only as placeholders***.

Rules for identifying and counting significant digits.
Digits that are **significant** are:

- All nonzero digits, such as in 563, which has 3 significant digits.
- All embedded zeros, such as in 7,500.06, which has 6 significant digits.
- Trailing zeros after the decimal point, such as in 2.4500, which has 5 significant digits.

Digits that are **not significant** are zeros merely used as place holders between other digits and the decimal point or isolated zeros in front of the decimal point:

- Leading zeros, such as in 0.000008, which has only 1 significant figure.
- Trailing zeros before the decimal point, such as in 16000, which has 2 significant digits.

Exercise:
Check Your Understanding

Problem:

Determine the number of significant figures in the following measurements:

a. 0.0009
b. 15,450.0
c. 87.990
d. 30.42
e. 2,500,000
f. 6.20×10^3

Solution:

(a) 1; the zeros in this number are placeholders that indicate the decimal point

(b) 6; here, the zeros indicate that a measurement was made to the 0.1 decimal point, so the zeros are significant

(c) 5; the final zero indicates that a measurement was made to the 0.001 decimal point, so it is significant

(d) 4; any zeros located in between significant figures in a number are also significant

(e) 2; the zeros here are merely placeholders

(f) 3; the zero here shows precision so should be counted, where as 6.2 $\times 10^3$ would only have 2 significant figures

Steps for rounding results of multiplication or division, using significant digits:

It is okay to have different units or labels, except for some circumstances, like calculating volume.
Multiply or divide the given numbers.
Round the result to the lowest count of significant digits from the given numbers.
Label the result correctly with proper units.

Example:
Exercise:

Problem:

The area of a circle can be calculated from its radius using $A = \pi r^2$.

Let's see how many significant figures the area has if the radius has only two—say, $r = 1.2$ m. Then,

Solution:
Equation:

$$A = \pi r^2 = (3.1415927...) \times (1.2\ \mathrm{m})^2 = 4.5238934\ \mathrm{m}^2$$

is what you would get using a calculator that has an eight-digit output. But because the radius has only two significant figures, it limits the calculated quantity to two significant figures or
Equation:

$$A{=}4.5\ \mathrm{m}^2,$$

even though π is good to at least eight digits.

Example:
Exercise:

Problem:

Multiply 458.3 cm × 0.002 cm. Round the answer using significant digits.

Solution:

Using a calculator, we get 458.3 × 0.002 = 0.9166. Since 0.002 only has one significant digit (all the zeros are placeholders), we must round the answer to 0.9. The final answer, including units, is 0.9 cm^2.

Example:
Exercise:

Problem:

Divide: 850 mL ÷ 60 min. Round the result using significant digits.

Solution:

Using the calculator, 850 ÷ 60 = 14.66666... Since 60 has only one significant digit, which is the lowest number of significant digits between the two numbers, so must the answer. Thus, the final answer, including units is 10 mL/min.

Note:
Try It

Exercise:

Problem:

Divide: $45.6\ yd^2 \div 9.1\ yd$. Round the result using significant figures.

Solution:

Using a calculator, $45.6 \div 9.1 = 5.010989011...$ Since 9.1 has the smallest number of significant digits, namely 2, so must the answer. Thus, the final answer, including units, is 5.0 yd.

Significant Figures in this Text

In this text, you only need to use these rules when specifically asked to do so. Furthermore, other examples and solutions to homework exercises will not necessarily follow these same rules. Whether you need to follow the rules of rounding with precision in your career will depend on your field.

Summary

- Accuracy of a measured value refers to how close a measurement is to the correct value.
- Precision of measured values is based on the last place value.
- The precision of a *measuring tool* is related to the size of its measurement increments. The smaller the measurement increment, the more precise the tool.
- Significant figures express the precision of a measuring tool.
- When adding or subtracting measured values, the final answer cannot contain more decimal places than the least precise value.
- When multiplying or dividing measured values, the final answer can contain only as many significant figures as the value with the least number of significant figures.

Homework

Express your answers to problems in this section to the correct number of significant figures or least precise place value and remember to include proper units.

Exercise:

Problem: Add 0.08 g and 132 mg.

Solution:

210 mg or 0.21 g

Exercise:

Problem: Find the difference: 4.05 L – 1.9 L.

Exercise:

Problem: Multiply: 6.1 m × 8 m × 5.02 m

Solution:

200 m^3

Exercise:

Problem: Multiply: 27 miles × 3.0 miles

Exercise:

Problem: Multiply: 12.6 kg × 0.25 hr.

Solution:

3.2 kg-hr.

Exercise:

Problem: Calculate: 56.4 m + 23 m – 10 m

Exercise:

Problem: Subtract: 1098 mi. – 940 mi.

Solution:

160 mi.

Exercise:

Problem: Subtract: 62.5 yd. − 8 yd.

Exercise:

Problem: 25 milligrams divided by 12 pills

Solution:

2.1 mg/pill

Exercise:

Problem: Multiply: (3.21 cm)(4.1 cm)